18. *Fourier Series. XIII. Transformation of Fourier Series*

By Miyoko KATAYAMA
Department of Mathematics, Hokkaidô University, Sapporo, Japan
(Comm. by Z. SUETUNA, M.J.A., Feb. 12, 1957)

Let $\Lambda=(\lambda_{\nu n})$ $(\nu, n=0,1,2,\cdots)$ be an infinite matrix whose elements are real numbers and let $f(x)$ be an integrable function periodic with period 2π, and its Fourier series be

$$(1) \qquad f(x)\sim\frac{1}{2}a_0+\sum_{\nu=1}^{\infty}(a_\nu \cos \nu x+b_\nu \sin \nu x).$$

The Fourier series (1) is said to be Λ-summable to $\alpha(x)$, if the series

$$(2) \qquad \alpha_n(x)=\frac{1}{2}a_0\lambda_{0n}+\sum_{\nu=1}^{\infty}\lambda_{\nu n}(a_\nu \cos \nu x+b_\nu \sin \nu x)$$

converges for all n and

$$(3) \qquad \lim_{n\to\infty}\alpha_n(x)=\alpha(x) \qquad \text{exists.}$$

If the convergence of both (2) and (3) is uniform in x, it is said to be uniformly Λ-summable to $\alpha(x)$.

Concerning the Λ-summability of Fourier series of all continuous functions, J. Karamata [1] has established the following

Theorem. *A necessary and sufficient condition that the Fourier series of all continuous functions be uniformly Λ-summable to $f(x)$ is that*

1) $$\lim_{n\to\infty}\lambda_{\nu n}=1 \qquad \textit{for every } \nu,$$

2) $$\int_0^\pi |K_{mn}(t)|\,dt\leqq M_n \qquad (n=0,1,2,\cdots),$$

where $K_{mn}(t)=\frac{1}{2}\lambda_{0n}+\sum_{\nu=1}^{m}\lambda_{\nu n}\cos \nu t$ *and* M_n *is independent of* m, *and*

3) $$\int_0^\pi |d\overline{K}_n(t)|=O(1) \qquad \textit{for all } n,$$

where

$$\overline{K}_n(x)=\lim_{m\to\infty}\int_0^x K_{mn}(t)dt.$$

In this note, we shall prove L_p-analogues $(p\geqq 1)$ of this theorem. For the proof we use the following theorems [2]:

Theorem A. *A necessary and sufficient condition that $\{\lambda_n\}$ be a sequence of uniform convergence factors of Fourier series of all functions belonging to L_p $(p>1)$, is that*

$$\int_{-\pi}^{\pi}|K_n(t)|^q dt=O(1) \qquad (n\to\infty),$$

where $K_n(t)=\lambda_0+\sum_{\nu=1}^{n}\lambda_\nu \cos \nu t$, *and* $\frac{1}{p}+\frac{1}{q}=1$.

Theorem B. *A necessary and sufficient condition that* $\{\lambda_n\}$ *be a sequence of uniform convergence factors of Fourier series of all functions belonging to* L, *is that*

$$K_n(t)=O(1)$$

uniformly in both n *and* t.

1. In the case $p>1$, we get the following

Theorem 1. *A necessary and sufficient condition that the Fourier series of all functions belonging to* L_p $(p>1)$ *be uniformly* Λ*-summable is that*

1) $\lim_{n\to\infty}\lambda_{\nu n}$ *exists for every* ν,

and

2) *there exist functions* $K_n(t)$ *belonging to* L_q *such that*

$$K_n(t)\sim\frac{1}{2}\lambda_{0n}+\sum_{\nu=1}^{\infty}\lambda_{\nu n}\cos \nu t$$

and

$$\int_{-\pi}^{\pi}|K_n(t)|^q dt=O(1) \qquad \textit{for every } n,$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Let $f(x)$ be a function in L_p and its Fourier series be

$$f(x)\sim\frac{1}{2}a_0+\sum_{\nu=1}^{\infty}(a_\nu \cos \nu x+b_\nu \sin \nu x).$$

By Theorem A, the series

$$\frac{1}{2}a_0\lambda_{0n}+\sum_{\nu=1}^{\infty}\lambda_{\nu n}(a_\nu \cos \nu x+b_\nu \sin \nu x)$$

converges uniformly for every n, if and only if

$$\int_{-\pi}^{\pi}|K_{\mu n}(t)|^q dt\leqq M_n, \tag{4}$$

where $K_{\mu n}(t)$ is the μth partial sum of the series

$$\frac{1}{2}\lambda_{0n}+\sum_{\nu=1}^{\infty}\lambda_{\nu n}\cos \nu t \tag{5}$$

and M_n is independent of μ. By (4), the series (5) is the Fourier series of a function $K_n(t)$ in L_q (cf. [3, p. 79]). Since $f(x)$ and $K_n(t)$ belong to the conjugate classes respectively, we get the Parseval formula (cf. [3, p. 88])

$$\frac{1}{\pi}\int_{-\pi}^{\pi}f(x+t)K_n(t)dt=\frac{1}{2}a_0\lambda_{0n}+\sum_{\nu=1}^{\infty}\lambda_{\nu n}(a_\nu \cos \nu x+b_\nu \sin \nu x).$$

We shall now consider the linear functionals on L_r:

$$A_n(f)=\frac{1}{\pi}\int_{-\pi}^{\pi}f(t)K_n(t)dt.$$

By a well-known theorem, those linear functionals $A_n(f)$ converge, if and only if

1) the norms of A_n are uniformly bounded, that is

$$\int_{-\pi}^{\pi}|K_n(t)|^q dt \leqq M \qquad (n=1,2,\cdots), \tag{6}$$

and

2) $\{A_n\}$ converges for every element of the base $\{\cos kx\}$, that is

$$A_n(\cos kx)=\lambda_{kn} \quad \text{converges.} \tag{7}$$

By the Riesz theorem [3, p. 153], (4) is a consequence of (6).

Thus we have proved that the conditions (6) and (7) are necessary and sufficient for the uniform convergence of (2) and the convergence of (3). It is easy to see the uniform convergence of (3) under the condition (6) (cf. [2]), and hence the theorem is proved.

2. In the case $p=1$, we get similarly the following

Theorem 2. *A necessary and sufficient condition that the Fourier series of all functions belonging to L be Λ-summable, is that*

1) $\lim_{n\to\infty}\lambda_{\nu n}$ *exists for every* ν,

2) $$|K_{\mu n}(t)|\leqq M_n,$$

and

3) *there exist bounded functions $K_n(t)$ such that*

$$K_n(t)\sim\frac{1}{2}\lambda_{0n}+\sum_{\nu=1}^{\infty}\lambda_{\nu n}\cos\nu t$$

and $|K_n(t)|\leqq M$ uniformly in both n and t.

Proof. Let $f(x)$ be a function in L with Fourier series (1). By Theorem B, the series

$$\frac{1}{2}a_0\lambda_{0n}+\sum_{\nu=1}^{\infty}\lambda_{\nu n}(a_\nu\cos\nu x+b_\nu\sin\nu x)$$

converges uniformly for every n, if and only if

$$|K_{\mu n}(t)|\leqq M_n. \tag{8}$$

By (8) the series

$$\frac{1}{2}\lambda_{0n}+\sum_{\nu=1}^{\infty}\lambda_{\nu n}\cos\nu t$$

is the Fourier series of a bounded and measurable function $K_n(t)$. Since $f(x)$ and $K_n(x)$ belong to the conjugate classes respectively, we get by a well-known theorem (cf. [3, p. 88]),

$$\frac{1}{\pi}\int_{-\pi}^{\pi}f(x+t)K_n(t)dt=\frac{1}{2}\lambda_{0n}a_0+\sum_{\nu=1}^{\infty}\lambda_{\nu n}(a_\nu\cos\nu x+b_\nu\sin\nu x).$$

Consider now the linear functionals on L:

$$A_n(f)=\frac{1}{\pi}\int_{-\pi}^{\pi}f(t)K_n(t)dt.$$

Then we may find similarly as in the proof of Theorem 1 the necessary and sufficient condition that $A_n(f)$ converges for every f, that is:

1) $$|K_n(t)| \leqq M$$

and

2) λ_{kn} converges as $n \to \infty$ for every k.

Thus we can complete the proof as in § 1.

3. Using above results, finally, we shall give some sufficient conditions for Fourier series of all functions belonging to L_p $(p>1)$ and L to be uniformly Λ-summable.

Corollary 1. *If a matrix* $\Lambda=(\lambda_{\nu n})$ *satisfy the following conditions*:

1) $\lim\limits_{n\to\infty} \lambda_{\nu n}$ *exists for every* ν,

2) $\lambda_{\nu n} \downarrow 0$ $(\nu \to \infty)$ *and* $\sum\limits_{\nu=0}^{\infty} \lambda_{\nu n}^q \nu^{q-2} < M$ *uniformly in* n,

then the Fourier series of all functions belonging to L_p $(p>1)$ *are uniformly* Λ*-summable.*

Corollary 2. *If a matrix* $\Lambda=(\lambda_{\nu n})$ *satisfy the following conditions*:

1) $\lim\limits_{n\to\infty} \lambda_{\nu n}$ *exists for every* ν,

2) $$\sum_{\nu=\mu}^{2\mu} |\lambda_{\nu n} - \lambda_{\nu+1,n}| = O\left(\frac{1}{\mu}\right) \quad (\mu \to \infty)$$

uniformly in n,

3) $\sum\limits_{\nu=0}^{\infty} \lambda_{\nu n} x^{\nu} = O(1)$ *uniformly in both* n *and* x,

then the Fourier series of all functions belonging to L *are uniformly* Λ*-summable.*

In fact, we can prove by the Hardy-Littlewood theorem [3, p. 213], that the condition 2) in Corollary 1 leads the condition 2) in Theorem 1, and using a theorem of Szász [4] slightly modified, that the conditions 2) and 3) in Corollary 2 lead the conditions 2) and 3) in Theorem 2. We shall omit their detailed proofs.

References

[1] J. Karamata and M. Tomić: Sur la sommation des séries de Fourier des fonctions continues, Acad. Serbe Sci. Publ. Inst. Math., **8** (1955).

[2] M. Katayama: Fourier series. VII. Uniform convergence factors of Fourier series, J. Fac. Sci. Hokkaidô Univ., Ser. I, **13**, no. 3 (1956).

[3] A. Zygmund: Trigonometrical series, Warszawa (1935).

[4] O. Szász: On uniform convergence of trigonometric series, Bull. Amer. Math. Soc., **50** (1944).