# 17. On Hardy and Littlewood's Theorem

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1. Let $f(x)$ be an $L$-integrable function with period $2\pi$, and its Fourier series be

(1) $$\frac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx).$$

A. Zygmund [1] has shown the following

**Theorem Z.** *If $f(x)$ belongs to Lip $\alpha$ where $0<\alpha\leqq 1$, then the series (1) is uniformly summable $(C, -\alpha+\delta)$ to $f(x)$ for every $\delta>0$.*

Later, Hardy and Littlewood [2] showed the following

**Theorem H., L.** *If $f(x)$ belongs to Lip $(\alpha, p)$ where $0<\alpha\leqq 1$ and $\alpha p>1$, i.e.*

$$\left(\int_0^{2\pi}|f(x+h)-f(x)|^p dx\right)^{1/p}=O(|h|^{\alpha})$$

*as $h\to 0$, then the series (1) is uniformly summable $(C, -\alpha+\delta)$ to $f(x)$ for every $\delta>0$.*

In this paper we shall improve the above theorem as follows:

**Theorem.** *If $f(x)$ is continuous in $(0, 2\pi)$, and belongs to Lip $(\alpha, 1/\alpha)$ where $0<\alpha\leqq 1$, i.e.*

$$\int_0^{2\pi}|f(x+h)-f(x)|^{1/\alpha}dx=O(h)$$

*as $h\to 0$, then the series (1) is uniformly summable $(C, -\alpha+\delta)$ to $f(x)$ for every $\delta>0$.*

2. The proof*) of our theorem is as follows. Let

$$\varphi(t)=\varphi_x(t)=f(x+t)+f(x-t)-2f(x),$$

then we have

(2) $$\varphi(t)\to 0 \text{ as } t\to 0 \text{ uniformly in } 0\leqq x\leqq 2\pi,$$

since $f$ is continuous.

We denote the $n$-th $(C,\gamma)$ mean of the series (1) by $\sigma_n^{\gamma}(x)$, then

$$\sigma_n^{-\alpha}(x)-f(x)=\frac{1}{\pi}\int_0^{\pi}\varphi(t)K_n^{-\alpha}(t)dt$$
$$=\frac{1}{\pi}\int_0^{K/n}+\frac{1}{\pi}\int_{K/n}^{\pi}=I_1+I_2$$

say, where $K_n^{\gamma}(t)$ is the $n$-th $(C,\gamma)$ Féjer kernel and

(3) $$|K_n^{-\alpha}(t)|\leqq\frac{n}{1-\alpha}+\frac{1}{2}\quad\text{for } 0\leqq t\leqq\pi,$$

*) The method of this proof has been suggested to me by Prof. G. Sunouchi.

and

(4) $$K_n^{-\alpha}(t)=\Re(e^{int}/A_n^{-\alpha}(1-e^{-it})^{1-\alpha})+O(1/nt^2)$$

for $0<t\leqq\pi$.

By (2) and (3) it holds

$$|I_1|<\varepsilon_n K$$

uniformly concerning $x$, where $\varepsilon_n>0$ and $\varepsilon_n\to 0$. And we see easily that, by (4), (2) and boundedness of $f$,

$$I_2=\Re\left(\frac{1}{2\pi A_n^{-\alpha}}\int_{K/n}^{\pi}\frac{\varphi(t)-\varphi(t+\pi/n)}{(1-e^{-it})^{1-\alpha}}e^{int}dt\right)+O(1/K^{1-\alpha}),$$

where $O$ is uniform concerning $x$.

Replacing $-\alpha$ by $-\alpha+\delta$ we have

(5) $$|\sigma_n^{-\alpha+\delta}(x)-f(x)|<C_1 n^{\alpha-\delta}\int_{K/n}^{\pi}\frac{|\varphi(t)-\varphi(t+\pi/n)|}{t^{1-\alpha+\delta}}dt + C_2/K^{1-\alpha+o}+\varepsilon_n K,$$

where, and in succession, $C$'s are absolutely positive constants, not depending on $x$.

First suppose that $\alpha<1$, then since $f\in \mathrm{Lip}(\alpha, 1/\alpha)$ we have

$$n^{\alpha-\delta}\int_{K/n}^{\pi}\frac{|\varphi(t)-\varphi(t+\pi/n)|}{t^{1-\alpha+\delta}}dt$$

$$\leqq n^{\alpha-\delta}\left(\int_0^{2\pi}|\varphi(t)-\varphi(t+\pi/n)|^{1/\alpha}dt\right)^{\alpha}\left(\int_{K/n}^{\pi}(1/t^{1-\alpha+\delta})^{1/(1-\alpha)}dt\right)^{1-\alpha}$$

$$\leqq C_3\, n^{\alpha-\delta}(1/n)^{\alpha}(n/K)^{\delta}=C_3/K^{\delta}.$$

In the case $\alpha=1$, since $f\in \mathrm{Lip}(1,1)$,

$$n^{1-\delta}\int_{K/n}^{\pi}\frac{|\varphi(t)-\varphi(t+\pi/n)|}{t^{\delta}}dt$$

$$\leqq n^{1-\delta}(n/K)^{\delta}\int_0^{2\pi}|\varphi(t)-\varphi(t+\pi/n)|dt$$

$$\leqq C_4 n^{1-\delta}(n/K)^{\delta}(1/n)=C_4/K^{\delta}.$$

Thus we have from (5)

$$|\sigma_n^{-\alpha+\delta}(x)-f(x)|<C_5/K^{\delta}+C_2/K^{1-\alpha+\delta}+\varepsilon_n K.$$

With $n\to\infty$ and then $K\to\infty$ we get the desired result.

## References

[1] A. Zygmund: Sur la sommabilité des séries de Fourier des fonctions vérifiant la condition de Lipschitz, Bulletin Acad. Cracovie, 1–9 (1925).

[2] G. H. Hardy and J. E. Littlewood: A convergence criterion for Fourier series, Math. Z., **28**, 612–634 (1928).