# 33. A Remark on Countably Compact Normal Space

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In [4], S. Kasahara, one of my colleagues, and I myself gave a new characterization of countably compact normal space. In this Note, we shall give a slight generalisation of Theorem 3 in [4]. Some writers ([2], [3], [5] and [6]) introduced the concepts of $\sigma$-discrete, $\sigma$-locally finite, and $\sigma$-star finite coverings of topological space, and they obtained the interesting results on some topological spaces. Let $\alpha$ be a family of open sets in a topological space $S$. $\alpha$ is said to be *discrete*, if every point of $S$ has a neighbourhood which meets at most one member of $\alpha$. $\alpha$ is said to be *point finite*, if every point of $S$ is contained in only finite many members of $\alpha$. $\alpha$ is said to be *star finite*, if every member of $\alpha$ meets only finite many members of $\alpha$. $\alpha$ is said to be *locally finite*, if every point of $S$ has a neighbourhood which meets only finite members of $\alpha$.

An open covering $\alpha$ is called *$\sigma$-discrete* (*$\sigma$-point finite*, *$\sigma$-star finite* or *$\sigma$-locally finite*), if $\alpha=\bigcup_{i=1}^{\infty}\alpha_i$ such that each $\alpha_i$ is discrete (point finite, star finite or locally finite). Then we shall prove the following

*Theorem 1. The following propositions of a normal space S are equivalent:*

1) *S is countably compact.*
2) *Every $\sigma$-point finite open covering has a finite subcovering.*
3) *Every $\sigma$-locally finite open covering has a finite subcovering.*
4) *Every $\sigma$-star finite open covering has a finite subcovering.*
5) *Every $\sigma$-discrete open covering has a finite covering.*

Proof. 2)$\rightarrow$1), 3)$\rightarrow$1) and 4)$\rightarrow$1) are obvious by Theorem 3 in [4]. 5)$\rightarrow$1) follows from the definition of countably compactness.

We must prove that (1) implies the other propositions (2), (3), (4) and (5). In general, (2)$\rightarrow$(3), (3)$\rightarrow$(4) are trivial.

First, we shall show (1)$\rightarrow$(2). Let $\alpha$ be a $\sigma$-point finite open covering of $S$, then there is a system of family $\alpha_n$ of open sets such that $\alpha=\bigcup_i\alpha_i$ and each $\alpha_i$ is point finite. Let $O_n$ be the sum of all members of $\alpha_n$, then $\beta=\{O_n\}$ is a countable open covering of $S$. Since $S$ is countably compact, $\beta$ has a finite covering $\{O_{n_1},\cdots,O_{n_k}\}$. Then the system $\gamma=\{\alpha_{n_1},\cdots,\alpha_{n_k}\}$ is an open covering and it is obvious that $\gamma$ is a point finite covering. Therefore, by Theorem 2 in [3], $\gamma$ has a finite covering. This shows (1)$\rightarrow$(2). Second, we shall show

(1)→(5) by a similar method. Suppose that $\alpha$ is a $\sigma$-discrete open covering of $S$. Then we have $\alpha=\bigcup_n \alpha_n$, where each $\alpha_n$ is a family of discrete open sets of $S$. Let $O_n$ be the sum of all members of $\alpha_n$. Then $\beta=\{O_n\}$ is a countable open covering of $S$, and $\beta$ has a finite covering $\{O_{n_1},\cdots,O_{n_k}\}$. Thus $\gamma=\{\alpha_{n_1},\cdots,\alpha_{n_k}\}$ is a locally finite open covering of $S$.

Hence, we can find a finite covering of $\gamma$ by Theorem 2 in [3]. Therefore we have a proof of Theorem 1.

*Any paracompact $T_1$-space is normal,* as shown by J. Dieudonné. Therefore we have Theorem 2, which contains a result by R. Arens and J. Dugundji [1, Th. 2.6]. It follows easily from Theorem 1.

*Theorem 2. The following statements of a paracompact space $S$ are equivalent*:

1) *$S$ is compact.*

2) *Every $\sigma$-discrete open covering has a finite subcovering.*

3) *Every $\sigma$-point finite* (*or point finite*) *open covering has a finite subcovering.*

4) *Every $\sigma$-locally finite* (*or locally finite*) *open covering has a finite subcovering.*

5) *Every $\sigma$-star finite* (*or star finite*) *open covering has a finite subcovering.*

6) *$S$ is countably compact.*

In their famous article, *Mémoire sur les espaces compacts*, Proc. Academy of Amsterdam, 1929, P. Alexandroff and P. Urysohn introduced the notion of *H-closed space,* and proved that a $T_2$-space $S$ is $H$-closed if and only if every open covering $\{O_\alpha\}$ has finite subfamily $\{O_{\alpha_i}\}$ such that $\bigcup \bar{O}_{\alpha_i}=S$. A covering $\alpha$ is said to be an *AU-covering,* if the covering $\alpha$ has a finite subfamily $\beta$ such that the closure of the union of the members of $\beta$ is $S$.

By the notion of *AU*-property, we shall prove the following

*Theorem 3. The following propositions of a normal space $S$ are equivalent*:

1) *$S$ is countably compact.*

2) *Every $\sigma$-discrete open covering is the AU-covering.*

3) *Every $\sigma$-point finite* (*or point finite*) *open covering is the AU-covering.*

4) *Every $\sigma$-locally finite* (*or locally finite*) *open covering has the AU-covering.*

5) *Every $\sigma$-star finite* (*or star finite*) *open covering is the AU-covering.*

Proof. By Theorem 1, it is clear that (1)→(2), (3), (4) and (5). On the other hand, the implication (3)→(4)→(5) is clear. Therefore, it is sufficient to prove 2)$\rightarrow$1) and 5)$\rightarrow$1). To prove 5)$\rightarrow$1), let $\alpha$ be a star finite open covering. Since $S$ is a normal space, $\alpha$ is shrinkable to a covering $\beta$. By 5), we can find a finite subfamily $\gamma$ such that the closure of the union of members of $\gamma$ is $S$. Therefore the family of finite members of $\alpha$ containing the members of $\gamma$ is the $AU$-covering. This shows 5)$\rightarrow$1).

To prove that (2) implies (1), let us suppose that $S$ is not countably compact, then there is an isolated set with denumerable elements $x_n$ $(n=1, 2, \cdots)$. Therefore, by the normality of $S$, we can find a family of disjoint open sets $O_n$ containing $x_n$. For each $n$, take open sets $O'_n, O''_n$ such that $O_n \supseteq O'_n \supseteq O''_n$, and let $O'_\infty = S - \bigcup \bar{O}''_n$, then $\alpha = \{O'_\infty, O'_1, O'_2, \cdots, O'_n, \cdots\}$ is a $\sigma$-discrete open covering. $\alpha$ is not $AU$-covering.

It is easily seen that $x_{n+1} \in \bigcup_{i=1}^{n} \bar{O}'_i$ and $x_{n+1} \bar{\in} \bar{O}'_\infty$, by $(S-\bigcup \bar{O}''_n) \frown O''_n = \phi$. Therefore we have $\bigcup_{i=1}^{n} \bar{O}'_i \cup \bar{O}'_\infty \neq S$ for any $n$. Hence we constructed a $\sigma$-discrete open covering which is no $AU$-covering.

Therefore, we have a proof of Theorem 3.

## References


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