30. Fourier Series. XV. Gibbs' Phenomenon

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1. Concerning Gibbs' phenomenon of the Fourier series H. Cramér [1] proved the following theorem.

Theorem 1. *There exists a number r_0, $0<r_0<1$, with the following property: If $f(x)$ is simply discontinuous at a point ξ, the (C, r) means $\sigma_n^r(x)$ of the Fourier series of $f(x)$ present Gibbs' phenomenon at ξ for $r<r_0$, but not for $r\geqq r_0$.*

On the other hand S. Izumi and M. Satô [2] proved the following theorems:

Theorem 2. *Suppose that $f(x)=a\psi(x-\xi)+g(x)$, where $\psi(x)$ is a periodic function with period 2π such that $\psi(x)=(\pi-x)/2$ $(0<x<2\pi)$, and where*

$$\limsup_{x\downarrow\xi} g(x)=0, \qquad \liminf_{x\uparrow\xi} g(x)=0,$$

$$\liminf_{x\downarrow\xi} g(x)\geqq -a\pi, \qquad \limsup_{x\uparrow\xi} g(x)\leqq a\pi,$$

$$\int_0^x |g(\xi+u)|\,du=o(|x|), \tag{1}$$

then Gibbs' phenomenon of the Fourier series of $f(x)$ appears at $x=\xi$.

Theorem 3. *In Theorem 2, if we replace the condition (1) by the following conditions:*

$$\int_0^x g(\xi+u)du=o(|x|),$$

and

$$\int_0^x \{g(t+u)-g(t-u)\}\,du=o(|x|)$$

uniformly for all t in a neighbourhood of ξ, then Gibbs' phenomenon of the Fourier series of $f(x)$ appears at $x=\xi$.

We proved that Theorem 1 holds even when the point ξ is the discontinuity point of the second kind, satisfying the condition in Theorem 2 [3]. More precisely,

Theorem 4. *Suppose that*

$$f(x)=a\psi(x-\xi)+g(x)$$

where $\psi(x)$ is a periodic function with period 2π such that

$$\psi(x)=(\pi-x)/2 \qquad (0<x<2\pi)$$

and where

$$\limsup_{x\downarrow\xi} g(x)=0, \qquad \liminf_{x\uparrow\xi} g(x)=0,$$

$$\liminf_{x\downarrow\xi} g(x)\geqq -a\pi, \qquad \limsup_{x\uparrow\xi} g(x)\leqq a\pi,$$

$$\int_0^x |g(\xi+u)|\,du=o(|x|).$$

Then there exists a number r_0, $0<r_0<1$ *with the following property: The* (C, r) *means of the Fourier series of* $f(x)$ *present Gibbs' phenomenon at* ξ *for* $r<r_0$, *but not for* $r\geqq r_0$, r_0 *being the Cramér number in Theorem 1.*

We shall extend Theorem 4 replacing the assumptions by those of Theorem 3. More precisely

Theorem 5. *Suppose that* $f(x)=a\psi(x-\xi)+g(x)$, *where* $\psi(x)$ *is a periodic function with period* 2π *such that* $\psi(x)=(\pi-x)/2$ $(0<x<2\pi)$, *and where*

$$\limsup_{x\downarrow\xi} g(x)=\liminf_{x\uparrow\xi} g(x)=0$$

$$\liminf_{x\downarrow\xi} g(x)\geqq -a\pi, \qquad \limsup_{x\uparrow\xi} g(x)\leqq a\pi$$

$$(2) \qquad \int_0^x g(\xi+u)du=o(|x|),$$

and

$$(3) \qquad \int_0^x \{g(t+u)-g(t-u)\}\,du=o(|x|),$$

uniformly for all t *in a neighbourhood of* ξ. *Then there exists a number* r_0, $0<r_0<1$, *with the following property: The* (C, r) *means of the Fourier series of* $f(x)$ *present Gibbs' phenomenon at* ξ *for* $r<r_0$, *but not for* $r\geqq r_0$, r_0 *being the Cramér number in Theorem 1.*

2. Proof of Theorem 5. Without loss of generality, we can suppose that $\xi=0$ and $a=1$. We have

$$\sigma_n^r(x, f)=\sigma_n^r(x, \psi)+\sigma_n^r(x, g).$$

By Theorem 1 $\sigma_n^r(\pi/n, \psi)$ tends to a constant which is greater than $\pi/2$ if $r<r_0$, but not greater than $\pi/2$ if $r\geqq r_0$. Since $\sigma_n^r(k\pi/n, \psi)$ is near to $\pi/2$ for sufficiently large k, if $r<r_0$, there is a k such that

$$(4) \qquad \tfrac{1}{2}\{\sigma_n^r(\pi/n, \psi)+\sigma_n^r(k\pi/n, \psi)\}$$

tends to a constant, greater than $\pi/2$; and if $r\geqq r_0$, then (4) tends to $\pi/2$. Hence it is sufficient to prove that $\sigma_n^r(\pi/n, g)+\sigma_n^r(k\pi/n, g)$ tends to zero as $n\to\infty$, for any r, $0<r<1$, and for any k.

Now

$$\sigma_n^r(x, g)=\frac{1}{\pi}\int_{-\pi}^{\pi} g(t+x)K_n^r(t)dt=\frac{1}{\pi}\int_{-\pi}^{\pi} g(t)K_n^r(t-x)dt,$$

where $K_n^r(t)$ is the nth Fejér kernel of order r. It is known that

$$(5) \qquad |K_n^r(t)|\leqq An$$

and

$$K_n^r(t)=\frac{1}{A_n^r}\frac{\sin\{(n+1/2+r/2)t-\pi r/2\}}{(2\sin t/2)^{r+1}}+\frac{r}{(n+1)(2\sin t/2)^2}$$
$$-\frac{1}{A_n^r}\sum_{\nu=n+1}^{\infty} A_{\nu+1}^{r-2}\frac{\sin\{(n-\nu)t-\pi/2\}}{(2\sin t/2)^2}$$

where $A_n^r=\binom{r+n}{n}$ [4]. We write

$$\sigma_n^r(x, g)=\frac{1}{\pi}\left(\int_0^{\pi}+\int_{-\pi}^{0}\right)g(t)K_n^r(t-x)dt=\frac{1}{\pi}(I+J).$$

We shall estimate I only, since J may be estimated quite similarly.

We set now

$$I=\int_0^{\pi/n}+\int_{\pi/n}^{\pi}=I_1+I_2.$$

Then by (5) and (2) we get

$$|I_1|=\left|\int_0^{\pi/n}g(t+x)K_n^r(t)dt\right|\leqq\left|\left[G(t)K_n^r(t)\right]_{t=0}^{t=\pi/n}\right|+\left|\int_0^{\pi/n}(K_n^r(t))'G(t)dt\right|=o(1),$$

where $G(t)=\int_0^t g(u+x)du$. Further

$$I_2=\int_{\pi/n}^{\pi}g(t+x)K_n^r(t)dt=\int_{\pi/n}^{\pi}g(t+x)\frac{\sin\{(n+1/2+r/2)t-\pi r/2\}}{A_n^r(2\sin t/2)^{r+1}}dt$$

$$+\int_{\pi/n}^{\pi}g(t+x)\frac{r}{(n+1)(2\sin t/2)^2}dt+\int_{\pi/n}^{\pi}g(t+x)\frac{1}{A_n^r}\sum_{\nu=n+1}^{\infty}A_{\nu+1}^r\frac{\cos(n-\nu)t}{(2\sin t/2)^2}dt$$

$$=I_3+I_4+I_5, \text{ say,}$$

and

$$I_3=\int_{\pi/n}^{\pi}g(t+x)\frac{\cos\{(n+(1+\nu)/2)t-(1+r)\pi/2\}}{A_n^r t^{1+r}}dt+I_7=I_6+I_7.$$

By the Riemann-Lebesgue theorem we easily see $I_7=o(1)$, and

$$I_6=\frac{1}{A_n^r}\int_{\pi/n}^{\pi}g(t+x)\cos\{(1+r)(t-\pi)/2\}\frac{\cos nt}{t^{1+r}}dt$$

$$-\frac{1}{A_n^r}\int_{\pi/n}^{\pi}g(t+x)\sin\{(1+r)(t-\pi)/2\}\frac{\sin nt}{t^{1+r}}dt=I_8-I_9, \text{ say.}$$

We have

$$I_9=\frac{1}{A_n^r}\int_{\pi/n}^{\pi}\chi(t)\frac{\sin nt}{t^{1+r}}dt=\frac{1}{A_n^r}\int_{\pi/n}^{2\pi/n}\left\{\sum_{k=0}^{n-2}(-1)^k\frac{\chi(t+k\pi/n)}{(t+k\pi/n)^{1+r}}\right\}\sin nt\,dt$$

where

$$\chi(t)=g(t+x)\sin\{(1+r)(t-\pi)/2\}.$$

By the second mean value theorem

$$|I_9|\leqq C\sum_{k=0}^{[(n-2)/2]}\frac{n}{k^{1+r}}\left|\int_{\pi/n}^{2\pi/n}\{g(x+t+2k\pi/n)-g(x+t-(2k-1)\pi/n)\}dt\right|+o(1)$$

which is $o(1)$ by (2). Similarly $I_8=o(1)$, and hence $I_3=o(1)$.

On the other hand

$$I_4=\int_{\pi/n}^{\pi}g(t+x)\frac{r}{(n+1)(2\sin t/2)^2}dt=\frac{A}{n+1}\int_{\pi/n}^{\pi}\frac{g(t+x)}{t^2}dt+o(1)$$

$$=\frac{A}{n+1}\left[\frac{G(t)}{t^2}\right]_{t=\pi/n}^{t=\pi}+\frac{A}{n+1}\int_{\pi/n}^{\pi}\frac{G(t)}{t^3}dt$$

$$=\frac{A}{n+1}\left\{\frac{G(\pi)}{\pi^2}-\left(\frac{n}{\pi}\right)^2 o\left(\frac{\pi}{n}\right)\right\}+\frac{A}{n+1}o\left(\int_{\pi/n}^{\pi}\frac{dt}{t^2}\right)=o(1).$$

And further

$$I_5=\int_{\pi/n}^{\pi} g(t+x)\frac{1}{A_n^r}\sum_{\nu=n+1}^{\infty}A_{\nu+1}^{r-2}\frac{\cos(n-\nu)t}{(2\sin t/2)^2}dt$$

$$=\frac{1}{A_n^r}\sum_{\nu=n+1}^{2n}A_{\nu+1}^{r-2}\int_{\pi/n}^{\pi}+\frac{1}{A_n^r}\sum_{\nu=2n+1}^{\infty}\int_{\pi/n}^{\pi}=I_{10}+I_{11}, \text{ say.}$$

Then

$$I_{10}=\frac{1}{A_n^r}\sum_{\lambda=1}^{n}A_{\lambda+n+1}^{r-2}\int_{\pi/n}^{\pi}g(t+x)\frac{\cos\lambda t}{(2\sin t/2)^2}dt$$

$$=\frac{1}{A_n^r}\int_{\pi/n}^{\pi}g(t+x)\frac{\sum_{\lambda=1}^{n}A_{\lambda+n+1}^{r-2}\cos\lambda t}{(2\sin t/2)^2}dt=\frac{1}{A_n^r}\left[G(t)\frac{\sum_{\lambda=1}^{n}A_{\lambda+n+1}^{r-2}\cos\lambda t}{(2\sin t/2)^2}\right]_{t=\pi/n}^{t=\pi}$$

$$-\frac{1}{A_n^r}\int_{\pi/n}^{\pi}G(t)\left(\frac{\sum_{\lambda=1}^{n}A_{n+\lambda+1}^{r-2}\cos\lambda t}{(2\sin t/2)^2}\right)'dt$$

where

$$\left(\frac{\sum_{\lambda=1}^{n}A_{n+\lambda+1}^{r-2}\cos\lambda t}{(2\sin t/2)^2}\right)'$$

$$=\frac{-(2\sin t/2)^2\sum_{\lambda=1}^{n}A_{n+\lambda+1}^{r-2}\lambda\sin\lambda t-4\sin t/2\cos t/2\sum_{\lambda=1}^{n}A_{n+\lambda+1}^{r-2}\cos\lambda t}{(2\sin t/2)^4}$$

Since $\lambda A_{n+\lambda+1}^{r-2}$ $(\lambda=1,2,\cdots,n)$ is monotone increasing, we have

$$\left|\sum_{\lambda=1}^{n}A_{n+\lambda+1}^{r-2}\lambda\sin\lambda t\right|\leqq\frac{A_{2n+1}^{r-2}\,n}{|\sin t/2|}$$

Also

$$\left|\sum_{\lambda=1}^{n}A_{n+\lambda+1}^{r-2}\cos\lambda t\right|\leqq\frac{A_{n+2}^{r-2}}{|\sin t/2|}$$

Hence

$$|I_{10}|\leqq\left|\frac{1}{A_n^r}\left\{G(\pi)\frac{\sum_{\lambda=1}^{n}A_{n+\lambda+1}^{r-2}(-1)^{\lambda}}{4}-G(\pi/n)\frac{\sum_{\lambda=1}^{n}A_{n+\lambda+1}^{r-2}\cos\lambda\pi/n}{(2\sin\pi/2n)^2}\right\}\right|$$

$$+\int_{\pi/n}^{\pi}|G(t)|\left\{\frac{nA_{2n+1}^{r-2}}{4|\sin t/2|^3}+\frac{|\cos t/2|A_{n+2}^{r-2}}{4|\sin t/2|^4}\right\}dt=o(1).$$

And further

$$I_{11}=\frac{1}{A_n^r}\sum_{\lambda=n+1}^{\infty}A_{\lambda+n+1}^{r-2}\int_{\pi/n}^{\pi}g(t+x)\frac{\cos\lambda t}{(2\sin t/2)^2}dt.$$

If we write

$$\int_{\pi/n}^{\pi}g(t+x)\frac{\cos\lambda t}{(2\sin t/2)^2}dt=\int_{\pi/n}^{\pi}g(t+x)\frac{\cos\lambda t}{t^2}dt+J,$$

then from the Riemann-Lebesgue theorem $J=o(1)$ as $\lambda\to\infty$. Now

$$\int_{\pi/n}^{\pi} g(t+x)\frac{\cos\lambda t}{t^2}\,dt$$
$$=\int_{\pi/n}^{\pi/n+\pi/\lambda}\sum_{k=0}^{l-1} g(x+t+k\pi/\lambda)\frac{(-1)^k\cos\lambda t}{(t+k\pi/\lambda)^2}\,dt+\int_{\pi/n+l\pi/\lambda}^{\pi} g(t+x)\frac{\cos\lambda t}{t^2}\,dt$$
$$=\sum_{k=0}^{[(l-1)/2]}\int_{\pi/n}^{\pi/n+\pi/\lambda}\left\{\frac{g(x+t+2k\pi/\lambda)}{(t+2k\pi/\lambda)^2}-\frac{g(x+t+(2k+1)\pi/\lambda)}{(t+(2k+1)\pi/\lambda)^2}\right\}\cos\lambda t\,dt+o(1).$$

In view of the second mean value theorem and (2), we get

$$\int_{\pi/n}^{\pi} g(t+x)\frac{\cos\lambda t}{t^2}\,dt=\sum_{k=0}^{l}\frac{o(1/\lambda)}{(\pi/n+k\pi/\lambda)^2}+o(1)$$
$$=o(1/\lambda)\sum_{k=[\lambda/n]}^{l+[\lambda/n]}\frac{1}{(k\pi/\lambda)^2}+o(1)=o(1/\lambda)\cdot\lambda^2\sum_{k=[\lambda/n]}^{\infty}\frac{1}{k^2}+o(1)$$
$$=o(n)+o(1).$$

Hence

$$|I_{11}|\leqq\frac{1}{A_n^r}\sum_{\lambda=n+1}^{\infty}|A_{\lambda+n+1}^{r-2}|\left|\int_{\pi/n}^{\pi} g(t+x)\frac{\cos\lambda t}{(2\sin t/2)^2}\,dt\right|$$
$$\leqq\frac{1}{n^\lambda}\sum_{\lambda=n+1}^{\infty}\frac{1}{(\lambda+n+1)^{2-r}}\,o(n)=\frac{1}{n^\lambda}\,\frac{1}{n^{1-\lambda}}o(n)=o(1).$$

Thus the theorem is proved.

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References

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