29. Fourier Series. XIV. Order of Approximation of Partial Sums and Cesàro Means

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1. It is well known that, if $f(t)$ belongs to the Lip α class, $0<\alpha<1$, then

(1) $$s_n(t,f)-f(t)=O(\log n/n^\alpha), \qquad \text{uniformly},$$

where $s_n(t,f)$ is the nth partial sum of the Fourier series of $f(t)$.

The factor $\log n$ on the right of (1) can not be replaced by the smaller. Then arises the problem when the factor $\log n$ may be omitted. As its answer it is known the following theorems.

Theorem 1.[1] *Let $0<\alpha<1$, $p>1$, $0<\beta<1$ and $\alpha=\beta-1/p$. If $f(t)$ belongs to the Lip (β, p) class which is a subclass of Lip α, then*

(2) $$s_n(t,f)-f(t)=O(1/n^\alpha), \qquad \text{uniformly}.$$

Theorem 2.[2] *If $f(t)$ belongs to the Lip α class, $0<\alpha<1$ and it is of monotonic type, then (2) holds.*

A function $f(t)$ is said to be of monotonic type, if there is a constant C such that $f(t)+Ct$ is monotonic in the infinite interval $(-\infty,\infty)$.

We shall here treat the following problem: If $f(t)$ belongs to the Lip α class or some other classes, then under what local condition

$$s_n(x,f)-f(x)=O(1/n^\alpha)$$

holds at a point x? Similar problem arises concerning Cesàro means. The latter was recently treated by T. M. Flett [4].

The theorems which we prove are as follows.

Theorem 3. *Let $0<\alpha<1$, $p>1$. Suppose that $f(t)$ belongs to the Lip α class, or Lip (α, p) class, or that $f(t)$ is of $(1/\alpha)$-bounded variation. If the function*

$$\theta(u)=u\varphi_x(u)=u\{f(x+u)+f(x-u)-2f(x)\}$$

is of bounded variation in the right neighbourhood of $u=0$ and

(3) $$\int_0^t |d\theta(u)|=O(t^{1+\alpha}),$$

then

(4) $$s_n(x,f)-f(x)=O(1/n^\alpha).$$

This contains Theorem 2 as a particular case. We can get also a corollary of Theorem 3 which contains Theorem 2 [see § 3].

In the case $\alpha=1$ in Theorem 3, it needs an additional condition that the integral

1) Cf. [1] and [2]. Lip (α, p) is a subclass of Lip $(\alpha-1/p)$.
2) Cf. [3].

$$\int_0^t \frac{|\varphi_x(u)|}{u^2} du$$

exists [see § 2].

Further we have

Theorem 4. *Let* $0<\alpha\leqq 1$. *If*

$$\int_0^h \{f(t+u)-f(t-u)\}\,du=O(h^{\alpha+1}/\log 1/h) \quad as\ h\to 0$$

uniformly in t, and if, for a fixed x,

$$\int_0^h \{f(x+u)-f(x)\}\,du=O(h^{1+\alpha}) \qquad as\ h\to 0,$$

then

$$s_n(x,f)-f(x)=O(1/n^\alpha).$$

This theorem may be proved by a slight modification of the proof of a theorem due to one of us [5, Theorem 7]. Hence the proof is omitted.

On the other hand, T. M. Flett [4] proved the following theorem:

Theorem 5. *Let* $0<\alpha<1$, $0<\delta\leqq\pi$. *If* $f(t)$ *is of bounded variation in the interval* $(-\delta,\delta)$ *and*

$$\int_0^t |d\varphi_x(u)|\leqq At^\alpha$$

when $0\leqq t\leqq\delta$, *then*

$$(5) \qquad \sigma_n^\alpha(x,f)-f(x)=O(1/n^\alpha),$$

where $\sigma_n^\alpha(x,f)$ *denotes the nth Cesàro mean of order* α *of the Fourier series of* $f(t)$.

We can generalize this in the following form.

Theorem 6. *Let* $0<\alpha<1$. *If* $\theta(u)=u\varphi_x(u)$ *is of bounded variation in the interval* $(0,\delta)$ *and*

$$(3) \qquad v(t)=\int_0^t |d\theta(u)|=O(t^{1+\alpha}) \qquad (0\leqq t\leqq\delta),$$

then (5) holds at x.

T. M. Flett has further proved that

Theorem 7. *Let* $0<\alpha<1$, $0\leqq\beta<1$, $0<\delta\leqq\pi$ *and* $k\geqq\alpha-\beta$. *If x is a point such that*

$$\rho_n(x)=O(1/n^\beta)$$

where $\rho_n(t)$ *is the nth term of the Fourier series of* $f(t)$, *and*

$$(6) \qquad \int_0^t |d\varphi_x(u)|=O(t^\alpha) \qquad (0\leqq t\leqq\delta),$$

then

$$\sigma_n^k(x)-f(x)=O(n^{-\alpha}).$$

We can generalize this as follows.

Theorem 8. *In Theorem 7, we can replace (6) by (3).*

2. Proof of Theorem 3. We follow, the proof of Young's test for convergence of Fourier series. We set $\varphi_x(u)=\varphi(u)$ and

$$v(t)=\int_0^t |d\theta(u)|.$$

We write

$$s_n(x,f)-f(x)=\frac{1}{\pi}\int_0^\pi \varphi_x(u)\frac{\sin(n+1/2)u}{2\sin u/2}du$$
$$=\frac{1}{\pi}\Big(\int_0^{1/n}+\int_{1/n}^\eta+\int_\eta^\pi\Big)=\frac{1}{\pi}(I_1+I_2+I_3),$$

where η is taken such that $\theta(n)$ is of bounded variation in the interval $(0,\eta)$. Since $\varphi(t)=O(t^\alpha)$, we have

$$|I_1|\leqq(n+1/2)\int_0^{1/n}|\varphi(u)|du=O(1/n^\alpha),$$

as $n\to\infty$. If $f(t)$ belongs to $\mathrm{Lip}\,\alpha$ or $\mathrm{Lip}\,(\alpha,p)$ or is of $(1/\alpha)$-bounded variation, then $I_3=O(1/n^\alpha)$ for a fixed η.

In order to estimate I_2, we set

$$\xi(t)=\varphi(t)/t=\theta(t)/t^2, \quad \text{for } t\in(1/n,\eta),$$
$$=0, \quad \text{otherwise}.$$

Since $\theta(t)$ is of bounded variation in the interval $(0,\eta)$, $\xi(t)$ is also there, and then $|I_2|\leqq V/n$ where V is the total variation of $\xi(t)$. Now let

$$1/n=t_0<t_1<t_2<\cdots<t_m=\eta,$$

then

$$\left|\frac{\theta(t_i)}{t_i^2}-\frac{\theta(t_{i-1})}{t_{i-1}^2}\right|\leqq|\theta(t_i)|\left|\frac{1}{t_i^2}-\frac{1}{t_{i-1}^2}\right|+\frac{|\theta(t_i)-\theta(t_{i-1})|}{t_{i-1}^2}$$

and hence, by (3)

$$V=\int_{1/n}^\eta|d\xi(u)|\leqq\int_{1/n}^\eta\frac{dv(u)}{u^2}+2\int_{1/n}^\eta\frac{|\theta(u)|}{u^3}du\leqq\frac{v(\eta)}{\eta^2}-\frac{v(1/n)}{(1/n)^2}+4An^{1-\alpha}$$
$$\leqq An^{1-\alpha}.$$

Thus we have $|I_2|\leqq A/n^\alpha$.

Collecting above estimations we get (4).

From the proof of Theorem 3, we get the following

Theorem 9. *If $f(x)$ belongs to the Lip 1 class and $\theta(u)=u\varphi_x(u)$ is of bounded variation in the right neighbourhood of $u=0$ and*

$$\int_0^t|d\theta(u)|=O(t^2), \qquad \int_0^\eta\frac{|\varphi_x(u)|}{u^2}du<\infty,$$

then

$$s_n(x,f)-f(x)=O(1/n).$$

3. If $f(t)$ is of monotonic type, then

$$f(t+h)-f(t)+Ch$$

increases or decreases for each t as h increases. Generalizing this idea, we say that $f(t)$ is of (α)-monotonic type $(0<\alpha<1)$ if there is a constant C such that

$$(7) \qquad f(t+h)-f(t)+Ch^\alpha$$

is monotonic for each t, where C is independent of t.

Functions of monotonic type can not have cusp, but functions of (α)-monotonic type may have cusps infinitely many. For example,

$$f(x)=\sum_{n=1}^{\infty}\cos nx/n^{1+\alpha}$$

belongs to the Lip α class, but not of monotonic type, since $f(x)\sim x^{\alpha}$ $(x\downarrow 0)$. Let

$$F(x)=\sum_{n=1}^{\infty}f(2^n x)/2^n,$$

then $F(x)$ is also a function of the Lip α class and of (α)-monotonic type, and further it has infinitely many cusps.

As a corollary of Theorem 3 we get

Theorem 10. *If $f(x)$ belongs to the Lip α class and is of (α)-monotonic type, $0<\alpha<1$, then (2) holds.*

For, we can suppose that (5) is monotone increasing. Hence

$$\int_0^t |d\theta(u)|\leqq\int_0^t |d\{u(f(x+u)-f(x))\}|$$
$$+\int_0^t |d\{u(f(x-u)-f(x))\}|=I_1+I_2,$$

and

$$I_1\leqq\int_0^t d\{u(f(x+u)-f(x)+Cu^{\alpha})\}+|C|\int_0^t du^{1+\alpha}$$
$$=t\{f(x+t)-f(x)+Ct^{\alpha}\}+|C|t^{1+\alpha}\leqq At^{1+\alpha},$$

where A is independent of x. Thus we get (2).

4. Proof of Theorem 6. We write

$$\sigma_n^{\alpha}(x,f)-f(x)=\frac{1}{\pi}\int_0^{\pi}\varphi_x(u)K_n^{\alpha}(u)du$$

where $K_n^{\alpha}(u)$ is the nth Fejér kernel of order α. It is known that

$$|K_n^{\alpha}(u)|\leqq An$$

and

$$K_n^{\alpha}(u)=\frac{1}{A_n^{\alpha}}\frac{\sin\{(n+1/2+\alpha/2)u-\pi\alpha/2\}}{(2\sin u/2)^{1+\alpha}}+\frac{\alpha}{n+1}\frac{1}{(2\sin u/2)^2}$$
$$+\frac{\theta}{n^2}\frac{8\alpha(1-\alpha)}{(2\sin u/2)^3},\qquad (|\theta|\leqq 1).$$

Setting $N=n+1/2+\alpha/2$,

$$\sigma_n^{\alpha}(x,f)-f(x)=\frac{1}{\pi}\int_0^{1/n}\varphi_x(u)K_n^{\alpha}(u)du$$
$$+\frac{1}{\pi}\frac{1}{A_n^{\alpha}}\int_{1/n}^{\pi}\varphi_x(u)\frac{\sin(Nu-\pi\alpha/2)}{(2\sin u/2)^{1+\alpha}}du+\frac{\alpha}{n+1}\int_{1/n}^{\pi}\frac{\varphi_x(u)}{(2\sin u/2)^2}du$$
$$+\frac{8\theta\alpha(1-\alpha)}{n^2}\int_{1/n}^{\pi}\frac{\varphi_x(u)du}{(2\sin u/2)^3}=I_1+I_2+I_3+I_4.$$

We shall prove that all I's are of order $O(1/n^{\alpha})$.

Since $\int_0^t d\theta(u)=t\varphi(t)=O(t^{1+\alpha})$, we get $\varphi(t)=O(t^\alpha)$ and then

$$I_1=O\left(n\int_0^{1/n} u^\alpha du\right)=O(n^{-\alpha}).$$

In order to estimate I_2 we write

$$I_2=\int_{1/n}^{\pi}=\int_{1/n}^{\delta}+\int_{\delta}^{\pi}=I_{21}+I_{22},$$

where

$$|I_{22}|\leqq\frac{A}{n^\alpha}\int_\delta^\pi\frac{|\varphi_x(u)|}{u^{1+\alpha}}du\leqq\frac{A}{n^\alpha}$$

and I_{21} is a linear combination of the nth Fourier coefficients of the function

$$\begin{aligned}\xi(t)&=\varphi_x(t)/(2\sin t/2)^{\alpha+1} &&\text{in } (\pi/n,\delta),\\ &=0 &&\text{otherwise.}\end{aligned}$$

$\xi(t)$ is of bounded variation and its total variation is

$$V=\int_{1/n}^{\delta}\frac{dv(u)}{u^{2+\alpha}}+2\int_{1/n}^{\delta}\frac{|\theta(u)|}{u^{3+\alpha}}du=\frac{v(\delta)}{\delta^{3+\alpha}}-\frac{v(1/n)}{(1/n)^{2+\alpha}}+An\leqq An.$$

Hence, by a well-known theorem,

$$\left|\int_{1/n}^{\delta}\frac{\varphi_x(u)}{(2\sin u/2)^{1+\alpha}}\sin(Nu-\pi\alpha/2)du\right|\leqq V/n\leqq A$$

and then $|I_{21}|\leqq A/n^\alpha$, from which it follows $|I_2|\leqq A/n^\alpha$.

Now

$$|I_3|\leqq\frac{A}{n}\int_{1/n}^{\pi}\frac{|\varphi_x(u)|}{u^2}du\leqq\frac{A}{n}\int_{1/n}^{\pi}\frac{du}{u^{2-\alpha}}\leqq\frac{A}{n^\alpha}.$$

Collecting above estimations we get the required relation (5).

5. Proof of Theorem 8. From the proof of Theorem 7, it is sufficient to prove that $I_2=O(1/n^\alpha)$. We get easily $I_{21}=O(1/n^\alpha)$. In order to prove $I_{22}=O(1/n^\alpha)$, it is sufficient to use a lemma due to T. M. Flett [4]:

Lemma. *Let* $-1<k\leqq 1$, *let*

$$\begin{aligned}\xi(t)&=1/(2\sin t/2)^{1+k} &&(\delta\leqq t\leqq\pi),\\ &=\mu t &&(0\leqq t<\delta),\end{aligned}$$

where $\mu\delta=1/(2\sin\delta/2)^{1+k}$ *and*

$$\rho_n(x)=O(1/n^\beta)$$

where $0\leqq\beta<1$. *Then*

$$\int_0^\pi\varphi_x(u)\xi(u)\sin nu\,du=O(1/n^\beta).$$

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