85. *On the Completion of the Ranked Spaces*

By Hatsuo OKANO

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1. In this note we shall consider the problem of completion:[1] construction of a complete ranked space[2] containing a given ranked space as a dense subspace.

Definition 1. Let R be a ranked space.[3] For a family F of fundamental sequences of R, we shall call a coordinate of F every neighbourhood $v(p)$ which is the first term of a fundamental sequence belonging to F. For two families F, G of fundamental sequences, $F \geq G$ means that, for every coordinate $v(p)$ of F, there exists a coordinate $u(q)$ of G such that $v(p) \supseteq u(q)$.

Definition 2. For a point p of R, $\mathfrak{S}(p)$ denotes the set of all fundamental sequences $u=\{u_\alpha(p_\alpha)\}$ such that $p_\alpha \equiv p$. Let R^* be the family of all families p^* of fundamental sequences satisfying the following conditions:

(1) If $u_\alpha=\{u^\alpha_\beta(p^\alpha_\beta);\ 0\leq\beta<\omega_{\mu_\alpha}\}$ $(0\leq\alpha<\gamma<\omega_\nu)$ belongs to p^* and $\lambda_\alpha(0\leq\lambda_\alpha<\omega_{\mu_\alpha})$ is an ordinal number, then there exists a member $u=\{u_\beta(p_\beta)\}$ of p^* such that $u_0(p_0)\subseteq \bigcap_\alpha u^\alpha_{\lambda_\alpha}(p^\alpha_{\lambda_\alpha})$.

(2) If $u=\{u_\alpha(p_\alpha)\}$, $v=\{v_\beta(q_\beta)\}\in p^*$, $u_0(p_0)\in\mathfrak{B}_{\gamma_0}$, $v_0(q_0)\in\mathfrak{B}_{\gamma'_0}$, $\gamma_0<\gamma'_0$ and $u_0(p_0)\supseteq v_0(q_0)$, then there exist a rank γ and $u(p_0)$ of rank γ such that $\gamma_0<\gamma\leq\gamma'$ and $u_0(p_0)\supseteq u(p_0)\supseteq v_0(q_0)$.

(3) $p^*\not\geq\mathfrak{S}(p)$ for any p except the case $p^*=\mathfrak{S}(p)$.

Then we obtain easily the following

Lemma 1. $\mathfrak{S}(p)$ satisfies the conditions (1) and (2). And, for an ω_ν-fundamental sequence $v=\{v_\alpha(p_\alpha);\ 0\leq\alpha<\omega_\nu\}$, let v^β denote the fundamental sequence $\{v_\alpha(p_\alpha);\ \beta\leq\alpha<\omega_\nu\}$ and v^* the set of such v^β, $0\leq\beta<\omega_\nu$. Then v^* satisfies the conditions (1) and (2).

Definition 3. For two members p^*, q^* of R^*, put $p^*\approx q^*$ if $p^*\geq q^*$ and $p^*\leq q^*$. By this equivalence relation, we shall classify R^* and denote this classification by $\hat{R}$. Let $W(V,\hat{p})$, where $\hat{p}$ is a point of $\hat{R}$ and V is a coordinate of some p^* belonging to $\hat{p}$, denote the set of all

1) Prof. K. Kunugi studied this problem in the notes "Sur les espaces complets et régulièrement complets. I–III", Proc. Japan Acad., **30**, 553–556, 912–916 (1954); **31**, 49–53 (1955).

2) See, for the notions and the terminologies, K. Kunugi, I., *Op. cit.*, H. Okano: Some operations on the ranked spaces. I, Proc. Japan Acad., **33**, 172–176 (1957) and H. Okano: On closed subspaces of the complete ranked spaces, Proc. Japan Acad., **33**, 336–337 (1957).

3) The rank of R is given by ω_ν. See K. Kunugi, I., *Op. cit.*

elements $\hat{q}$ of $\hat{R}$ such that, for some coordinates U of q^* of $\hat{q}$, $I\{U\}$[4] $\subseteq V$. Take all $W(V, \hat{p})$ for the neighbourhoods of $\hat{p}$ in $\hat{R}$. Then F. Hausdorff's axiom (A)[5] is satisfied.

Lemma 2. $\omega(\hat{R}) \geq \omega_\nu$.

Proof. For any point $\hat{p}$ of $\hat{R}$ and any sequence of neighbourhoods $W(V_0, \hat{p}) \supseteq W(V_1, \hat{p}) \supseteq \cdots \supseteq W(V_\alpha, \hat{p}) \supseteq \cdots$, $0 \leq \alpha < \gamma < \omega_\nu$ of $\hat{p}$, there exists $p_\alpha^* \in \hat{p}$, for any α, such that V_α is a coordinate of p_α^*: there exists a fundamental sequence $u_\alpha = \{u_\beta^\alpha(p_\beta^\alpha);\ 0 \leq \beta < \omega_{\mu_\alpha}\} \in p_\alpha^*$ such that $u_0^\alpha(p_0^\alpha) = V_\alpha$. Since $p_\alpha^* \approx p_0^*$ for each α, then, by the condition (1), there exists a fundamental sequence u of p_0^* whose first term U is contained in $\bigcap_\alpha V_\alpha$. So $W(U, \hat{p}) \subseteq \bigcap_\alpha W(V_\alpha, \hat{p})$ and, hence, $\omega(\hat{R}, \hat{p}) \geq \omega_\nu$ for any $\hat{p}$ of $\hat{R}$. And consequently $\omega(\hat{R}) \geq \omega_\nu$.

Definition 4. We shall give a rank to $\hat{R}$. Choose a representative p^* from each $\hat{p}$ but, if $\hat{p} \ni \mathfrak{S}(p)$, we shall choose $\mathfrak{S}(p)$. Put $\mathfrak{B}_\alpha (0 \leq \alpha < \omega_\nu)$ = the set of every $W(V, \hat{p})$ such that V is of rank α and a coordinate of a representative of a point. Then axiom (a)[6] is satisfied and $\hat{R}$ is a ranked space.

2. We shall, hereafter, assume the following axioms for R.

Axiom (T_1'). For any two distinct points p and q, there exists a neighbourhood $v(p)$ of p and $u(q)$ of q such that $q \notin I\{v(p)\}$ and $p \notin I\{u(q)\}$.

Axiom (C'). If a point q is contained in a neighbourhood $v(p)$, then there exists a neighbourhood $u(q)$ of q such that $I\{u(q)\} \subseteq v(p)$.

By axiom (T_1'), $\mathfrak{S}(p) \in R^*$ for every p of R. We shall denote by $\varphi(p)$ the element of $\hat{R}$ containing $\mathfrak{S}(p)$.

Lemma 3. In $\hat{R}$, for any ω_ν-fundamental sequence $W = \{W_\alpha(\hat{p}_\alpha);\ 0 \leq \alpha < \omega_\nu\}$, we have $\bigcap_\alpha I\{W_\alpha(\hat{p}_\alpha)\} \neq 0$.

Proof. Let

$W(V_0, \hat{p}_0) \supseteq \cdots \supseteq W(V_\alpha, \hat{p}_\alpha) \supseteq \cdots$, $0 \leq \alpha < \omega_\nu$, $W(V_\alpha, p_\alpha) \in \mathfrak{B}_{\gamma_\alpha}$, be a fundamental sequence in $\hat{R}$. Then for each α there are a representative p_α^* of $\hat{p}_\alpha$ and a fundamental sequence $u_\alpha = \{u_\beta^\alpha(p_\beta^\alpha)\}$ of R contained in p_α^* such that $u_0^\alpha(p_0^\alpha) = V_\alpha$. Then, by axiom ($C'$),

$$u_0^0(p_0^0) \supseteq u_0^1(p_0^1) \supseteq \cdots \supseteq u_0^\alpha(p_0^\alpha) \supseteq \cdots .$$

And, by the condition (2), for each α, there exist a rank $\gamma'_{2\alpha}$ and $w_{2\alpha}(p_0^{2\alpha})$ of rank $\gamma'_{2\alpha}$ such that $\gamma_{2\alpha} < \gamma'_{2\alpha} \leq \gamma_{2\alpha+1}$ and $u_0^{2\alpha}(p_0^{2\alpha}) \supseteq w_{2\alpha}(p_0^{2\alpha}) \supseteq u_0^{2\alpha+1}(p_0^{2\alpha+1})$. Put

4) For a subset A, $I\{A\}$ denotes the *interior* of A: $p \in I\{A\}$ if and only if there exists a neighbourhood $v(p)$ of p such that $v(p) \subseteq A$.

5) F. Hausdorff: Grundzüge der Mengenlehre, 213 (1914).

6) See K. Kunugi, I., *Op. cit.*, Définition 2.

$$q_\alpha=\begin{cases} p_0^\alpha & \text{if } \alpha \text{ is even} \\ p_0^{\alpha-1} & \text{if } \alpha \text{ is odd,}\end{cases} \qquad v_\alpha(q_\alpha)=\begin{cases} u_0^\alpha(p_0^\alpha) & \text{if } \alpha \text{ is even} \\ w_{\alpha-1}(p_0^{\alpha-1}) & \text{if } \alpha \text{ is odd.}\end{cases}$$

Then $v=\{v_\alpha(q_\alpha);\ 0\leq\alpha<\omega_\nu\}$ is a fundamental sequence of R. If $v^*\geq\mathfrak{S}(p)$ for some p, then $\varphi(p)\in\bigcap_\alpha I\{W(V_\alpha,\hat{p}_\alpha)\}$. If $v^*\ngeq\mathfrak{S}(p)$ for every p, then, by Lemma 1, $v^*\in R^*$. Let $\hat{v}$ be the class which contains v^*, then $\hat{v}\in\bigcap_\alpha I\{W(V_\alpha,\hat{p}_\alpha)\}$.

Theorem 1. *If, for any fundamental sequence* $u=\{u_\alpha(p_\alpha);\ 0\leq\alpha<\omega_\mu\}$ *such that* $\omega_\mu<\omega_\nu$, *we have* $\bigcap_\alpha I\{u_\alpha(p_\alpha)\}\neq 0$ *in* R,[7] *then* $\hat{R}$ *is complete.*

Proof. Let $\{W(V_\alpha,\hat{p}_\alpha);\ 0\leq\alpha<\omega_\mu\}$, be a fundamental sequence in $\hat{R}$. If $\omega_\mu=\omega_\nu$. $\bigcap_\alpha I\{W(V_\alpha,\hat{p}_\alpha)\}\neq 0$ by Lemma 3. If $\omega_\mu<\omega_\nu$, we can easily verify that $\bigcap_\alpha I\{V_\alpha\}$ contains at least a point of R, say p. Then $\varphi(p)\in\bigcap_\alpha I\{W(V_\alpha,\hat{p}_\alpha)\}$.

Theorem 2. $\varphi(R)$[8] *is dense in* $\hat{R}$ *for the both topologies:*[9] $\overline{\varphi(R)}=\widetilde{\varphi(R)}=\hat{R}$.

Proof. For any point $\hat{p}$ of $\hat{R}$ and any neighbourhood $W(V,\hat{p})$ of $\hat{p}$, there exists a fundamental sequence $u=\{u_\alpha(p_\alpha)\}$ of R such that $V=u_0(p_0)$. Then we have $\varphi(p_0)\in W(V,\hat{p})$ and consequently $\overline{\varphi(R)}=\hat{R}$. Let $\hat{p}$ be any point of $\hat{R}$, p^* an element of $\hat{p}$ and $v=\{v_\alpha(p_\alpha)\}$ a fundamental sequence of p^*. Since $v_\alpha(p_\alpha)$ is a coordinate of $\varphi(p_\alpha)$ and $\hat{p}\in\bigcap_\alpha I\{W(v_\alpha(p_\alpha),\ \varphi(p_\alpha))\}$, then we have $\widetilde{\varphi(R)}=\hat{R}$.

Theorem 3. *The mapping* φ: $p\to\varphi(p)$ *is one-to-one and bi-continuous for the both topologies.*

Proof. (i) φ is *one-to-one*: let p, q be two distinct points of R, then, by axiom (T_1'), there exist $u(p)$ and $v(q)$ such that $p\notin I\{v(q)\}$ and $q\notin I\{u(p)\}$. Hence $\varphi(p)\neq\varphi(q)$.

(ii) φ is *bi-continuous*: it results from the fact that $\varphi(v(p))=W(v(p),\varphi(p))$ and, for each $p^*\in\varphi(p)$, $p^*\geq\mathfrak{S}(p)$.

3. Remark 1. If R is a metric space, then the completion $\hat{R}$ in our sense coincides with the classical one.

Remark 2. The hypothesis of Theorem 1 is satisfied if $\omega_\nu=\omega_0$ in R.

Remark 3. We shall denote by $\omega^*(R)$ the depth of R in T. Shirai's sense (T. Shirai: A remark on the ranked space. II, Proc. Japan Acad., 33, 139–142). If $\omega^*(R)\geq\omega_\nu$, then the hypothesis of Theorem 1 is satisfied.

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7) See Remarks 1 and 2 of Section 3.

8) $\varphi(R)$ denotes the set of all points $\hat{p}$ of the form $\hat{p}=\varphi(p)$, where $p\in R$.

9) See H. Okano: On closed subspaces of the complete ranked spaces, *Op. cit.*