89. A Theorem on Continuous Convergence

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(Comm. by K. KUNUGI, M.J.A., July 12, 1957)

A few years ago some interesting results on continuous convergence were obtained by C. Kuratowski [3], R. Arens and J. Dugundji [1]. Following H. Hahn [2], we shall define it as follows: A sequence of real valued functions $f_n(x)$ on a topological space S *converges continuously* to $f(x)$ on S if and only if $f_n(x_n)\to f(x)$ whenever $x_n\to x$ on S. In his "General Topology", W. Sierpiński [4, pp. 156–158] has proved that *a metric space M is compact if and only if the continuously convergence on M implies uniformly convergence on M.* Recently, S. Stoilow [5] proved a theorem on continuous convergence. In this Note, we shall generalize the theorem mentioned above of W. Sierpiński. To do so, we shall first prove the following

Theorem 1. If $f_n(x)$ converges continuously to $f(x)$ on a sequentially compact space S, the convergence is uniform.

By a *sequentially compact space,* we shall mean every sequence has a convergent subsequence.

Following the method of S. Stoilow [5, pp. 247–248], if $f_n(x)$ on any topological space S converges continuously to $f(x)$, we can prove that $f(x)$ is sequentially continuous: $x_n\to x_0$ implies $f(x_n)\to f(x_0)$. To prove Theorem 1, suppose that $f_n(x)$ is not convergent uniformly to $f(x)$. Then we can find a positive ε and an infinite sequence x_n such that

$$|f_{m_n}(x_n)-f(x_n)|>\varepsilon. \qquad (n=1,2,\cdots)$$

Since S is sequentially compact, there is a convergent subsequence x_{n_i} of x_n. Let x_0 be its limit point, then we have $f(x_{n_i})\to f(x_0)$. On the other hand, since $f_{m_{n_i}}(x)$ converges continuously to $f(x)$, we have $f_{m_{n_i}}(x_{n_i})\to f(x_0)$, which contradicts $|f_{m_{n_i}}(x_{n_i})-f(x_{n_i})|>\varepsilon$. This completes the proof.

Next, we shall prove the following

Theorem 2. If the continuous convergence of $f_n(x)$ on a completely regular space S to $f(x)$ on S implies the uniformly convergence, then S is countably compact.

Proof. Suppose that S is not countably compact, then there is a countable set a_n without cluster point. Therefore for each a_n, there is a neighbourhood U_n of a_n such that $U_m\frown U_n=\phi$ for $m\neq n$. Since S is completely regular, we can find a continuous function $f_n(x)$ such that

$$f_n(x)=\begin{cases} 0 & x \overline{\in} U_n \\ 1 & x=a_n \end{cases}$$

and $0 \leq f_n(x) \leq 1$ on S. Let $x_n \to x$, then, for sufficiently large n, we have $f_n(x_n)=0$ by the definition of $f_n(x)$ and $\{a_n\}$. Thus $f_n(x)$ converges continuously to 0. On the other hand, by the definition of $f_n(x)$, $f_n(x)$ is not uniformly convergence. This completes the proof.

From Theorems 1 and 2, we have the following

Theorem 3. For a weak separable completely regular space S, the following conditions are equivalent:

1) *S is sequentially compact.*

2) *S is countably compact.*

3) *Continuously convergence on S implies uniformly convergence.*

By using the method of the proof of Theorem 1, we can prove the following

Theorem 4. In a sequentially compact space S, if a sequence of continuous functions $f_n(x)$ converges uniformly to a continuous function $f(x)$ on every compact set of S, then $f_n(x)$ converges uniformly to $f(x)$ at every point of S.

For the definition of uniformly convergent at one point, see H. Hahn [2, p. 214].

Proof. Suppose that there is a point a such that $f_n(x)$ does not converge uniformly to $f(x)$ at a. Then there are a positive number ε and indices k_n and a sequence of points a_n such that

(1) $$|f_{k_n}(a_n)-f(a_n)|>\varepsilon. \quad (n=1, 2, \cdots)$$

Since $\{a_n\}$ contains a convergent subsequence $\{a_{n_p}\}$, let a_0 be the limit point of it, and we shall consider $C=\{a_0\} \smile \{a_{n_p} \mid p=1, 2, \cdots\}$. The set C is compact and $f_n(x)$, $f(x)$ are continuous, therefore $f_n(x)$ converges uniformly to $f(x)$ on C, which contradicts (1). Therefore, the proof is complete.

References

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