114. On B-covers and the Notion of Independence in Lattices

By Yataro MATSUSHIMA
Gunma University, Maebashi

(Comm. by K. KUNUGI, M.J.A., Oct. 12, 1957)

Introduction. In [3], L. M. Kelley has introduced the concept of B-covers as metric-between in a normed lattice. We have extended this notion to the case of general lattices in [4] and studied the geometries in lattices by means of B-covers and B^*-covers in [5]. In the first section of this paper we shall show that the relation "relative modularity" or "relative independence" which is derived from Wilcox [1] has a close connection with the J-cover or the CJ-cover which is a part of the B-cover in lattices. In the second section we shall consider the relations between the B-covers and independent sets in lattices.

For any two elements a, b of a lattice L, we shall define as follows.

$J(a,b)=\{x\,|\,(a\frown x)\smile(b\frown x)=x,\ x\in L\}$, $CJ(a,b)=\{x\,|\,(a\smile x)\frown(b\smile x)=x,\ x\in L\}$. $J(a,b)$ is called the J-cover of a and b, and if $x\in J(a,b)$, then we shall write $J(axb)$. Similarly we shall define CJ-cover and $CJ(axb)$.

$B(a,b)=J(a,b)\frown CJ(a,b)$ is called the B-cover of a and b and we shall write axb when $x\in B(a,b)$ (cf. [4, 5]).

1. Relative modular pairs and J-covers (CJ-covers). Following L. R. Wilcox [1], (a,b) is called a modular pair when $x\leqq b$ implies $(x\smile a)\frown b=x\smile(a\frown b)$, and in this case we write $(a,b)M$. In [5] we defined a relative modular pair $(a,b)M^*$ to be a pair (a,b) such that $a\frown b\leqq x\leqq b$ implies $(x\smile a)\frown b=x\smile(a\frown b)$.

B-covers treat "between" in lattices (cf. [4, 5]), while J-covers and CJ-covers may be considered as describing "semi-between" in lattices.

In the following L is always assumed to be a lattice.

Lemma 1.1. *The following statements are equivalent in case* $b'\leqq b$:

(a) $(b'\smile a)\frown b=b'\smile(a\frown b)=b$. $((b'\smile a)\frown b=b'\smile(a\frown b)=b')$.

(b) $J(abb')$ $(CJ(ab'b))$.

Proof. If $(b'\smile a)\frown b=b'\smile(a\frown b)=b$, then we have $(a\frown b)\smile(b\frown b')=(a\frown b)\smile b'=b$, that is $J(abb')$. Conversely if $J(abb')$, then we have $b=(a\frown b)\smile(b\frown b')\leqq b\frown(a\smile b')\leqq b$, and hence we have $(b'\smile a)\frown b=b=b'\smile(a\frown b)$. Similarly we can treat the other case.

Theorem 1.1. *If* $J(abb')$ *(resp.* $CJ(ab'b)$*) holds for any* b' *with* $b'\leqq b$ *then we have* $(a,b)M$.

Proof. It is obvious from Lemma 1.1.

Remark. $(a, b)M$ does not necessarily imply that either $J(abb')$ or $CJ(ab'b)$ holds for any $b' \leqq b$.

Indeed if $b \geqq a \geqq b'$, then $(a, b)M$ but neither $J(abb')$ nor $CJ(ab'b)$ since $(a \frown b) \smile (b \frown b') = a \smile b' = a$, $(a \smile b') \frown (b' \smile b) = a \frown b = a$.

Corollary 1.1. *For* $b' \leqq b$, bab' *implies* $(b' \smile a) \frown b = b' \smile (a \frown b) = a$ *and vice versa.*

Proof. If bab', then we have $b \frown b' \leqq a \leqq b \smile b'$ by [4, Lemma 1], and hence $b' \leqq a \leqq b$. Thus we have $(b' \smile a) \frown b = b' \smile (a \frown b) = a$.

Conversely if $(b' \smile a) \frown b = b' \smile (a \frown b) = a$, then we have $b' \leqq a$ from $b' \smile (a \frown b) = a$, and $a \leqq b$ from $(b' \smile a) \frown b = a$. Hence we have $b' \leqq a \leqq b$, thus we have bab'.

Lemma 1.2. *For* $b' \leqq b$, $(b' \smile a) \frown b = b' \smile (a \frown b) = x$ *implies* $J(axb')$ *and* $CJ(axb)$.

Proof. By hypothesis, we have $b' \leqq x \leqq b$, and hence $(a \smile x) \frown (b \smile x) = (a \smile x) \frown b = (a \smile b' \smile (a \frown b)) \frown b = (a \smile b') \frown b = x$, that is $CJ(axb)$. Similarly $(a \frown x) \smile (b' \frown x) = (a \frown x) \smile b' = b' \smile (a \frown b \frown (b' \smile a)) = b' \smile (a \frown b) = x$ by hypothesis; thus we have $J(axb')$.

Remark. For $b' \leqq b$, $CJ(axb)$ and $J(axb')$ do not necessarily imply $(b' \smile a) \frown b = b' \smile (a \frown b) = x$.

For instance, if L contains 9 elements $a, b, a', b', a_1, b_1, e, f, x$ such that $f > b > b' > b_1 > e$, $f > a' > a > a_1 > e$, $a' \frown b = x = a_1 \smile b_1$, then we have $CJ(axb)$, $J(axb')$ but $(b' \smile a) \frown b = b \neq x$.

Lemma 1.3. *If* $b' \smile (a \frown b)$ *belongs to* $CJ(a, b)$ *for every* b' *such that* $b' \leqq b$, *then we have* $(a, b)M$.

Proof. We have $(a \smile b' \smile (a \frown b)) \frown (b \smile b' \smile (a \frown b)) = b' \smile (a \frown b)$ by hypothesis, and hence $(b' \smile a) \frown b = b' \smile (a \frown b)$ for $b' \leqq b$, that is $(a, b)M$.

Lemma 1.4. *If* $(b' \smile a) \frown b$ *belongs to* $J(a, b')$ *for every* b' *such that* $b' \leqq b$, *then we have* $(a, b)M$.

Proof. By hypothesis, we have $(a \frown (b' \smile a) \frown b) \smile (b' \frown (b' \smile a) \frown b) = (b' \smile a) \frown b$, and hence $(a \frown b) \smile b' = (b' \smile a) \frown b$ for $b' \leqq b$, that is $(a, b)M$.

Theorem 1.2. *In* L, *the following statements are equivalent:*

(a) $b' \smile (a \frown b) \in CJ(a, b)$ *holds for every* b' *with* $b' \leqq b$.

(b) $(b' \smile a) \frown b \in J(a, b')$ *holds for every* b' *with* $b' \leqq b$.

(c) $(a, b)M$.

Proof. It follows from Lemmas 1.2, 1.3 and 1.4.

Remark. $J(a, b') \ni b' \smile (a \frown b)$ for any b' with $b' \leqq b$ does not necessarily imply $(a, b)M$.

For if L contains 5 elements a, b, b', e, f such that $f > b > b' > e$, $f > a > e$, $a \smile b = a \smile b' = f$, $a \frown b = a \frown b' = e$, then $b' \smile (a \frown b) = b'$ belongs to $J(a, b')$, but $(a, b)M$ does not hold.

Theorem 1.3. *If every element* b' *such that* $a \frown b \leqq b' \leqq b$ *belongs to* $CJ(a, b)$, *then we have* $(a, b)M^*$ *and vice versa.*

Proof. We have $(b'\smile a)\frown b=(b'\smile a)\frown(b'\smile b)=b'$ by $CJ(ab'b)$, and hence $b'\frown(a\smile b)=(b'\smile a)\frown b$ for $a\frown b\leqq b'\leqq b$, thus we have $(a, b)M^*$. Conversely if $(a, b)M^*$, then $(a\smile b')\frown(b\smile b')=(a\smile b')\frown b=b'\smile(a\frown b)=b'$ for $a\frown b\leqq b'\leqq b$, hence we have $CJ(ab'b)$.

Theorem 1.4. *In L, $(a, b)M$ is equivalent to $(a, b)M^*$.*

Proof. Since $(a, b)M$ implies $(a, b)M^*$, we have only to prove that $(a, b)M^*$ implies $(a, b)M$. Assume that $CJ(a, b)$ contains every b' such that $a\frown b\leqq b'\leqq b$; then $b''\smile(a\frown b)$ belongs to $CJ(a, b)$ for any $b''\leqq b$ since $a\frown b\leqq b''\smile(a\frown b)\leqq b$. Accordingly we have $(a, b)M$ by Lemma 1.3.

Theorem 1.4 is obtained in ②, (2), § 4 in [5].

2. Independence. In this section we shall use the notations and lemmas obtained by L. R. Wilcox [1] and G. Birkhoff [2].

Definition. $(a, b)\perp$ means that $a\frown b=0$, $(a, b)M$.

Definition. We write $(a_1, a_2, \cdots, a_n)\perp$ if $(\sum(a_i;\ i\in S),\ \sum(a_i;\ i\in T))\perp$ for every $S, T\subset[1, 2, \cdots, n]$ for which $j\in S$, $k\in T$ implies $j<k$.

Lemma 2.1. If $(a, b)\perp$, $a'\leqq a$, $b'\leqq b$ imply $(a', b')\perp$.

Lemma 2.2. If $(a, b)M$ and $(c, a\smile b)M$, $c\frown(a\smile b)\leqq a$, then $(c\smile a, b)M$ and $(c\smile a)\frown b=a\frown b$.

Lemma 2.3. If $(a, b)M$ and $c\leqq b$, then $(c\smile a, b)M$.

Lemma 2.4. Let $n=1, 2, \cdots$ and $a_1, a_2, \cdots, a_n$ be given. Then $(a_1, \cdots, a_n)\perp$ if and only if $(a_i, a_{i+1}\smile\cdots\smile a_n)\perp$ for $i=1, 2, \cdots, n-1$.

Definition. We write $(a_1, \cdots, a_n)\perp_S$ if $(a_{j_1}, a_{j_2}, \cdots, a_{j_n})\perp$ for every permutation $i\to j_i$ of the set of integers $[1, 2, \cdots, n]$.

Lemma 2.5. A lattice of finite length is semi-modular if and only if the relation of modularity between pairs of elements of L is symmetric.

Lemma 2.6. Let L be a semi-modular lattice of finite length; then $(a_1, a_2, \cdots, a_n)\perp$ implies $(a_1, a_2, \cdots, a_n)\perp_S$.

Now we shall define relative independence; we shall write $(a, b)\perp_p$ if $a\frown b=p$, $(a, b)M^*$. Then we have the next theorem.

Theorem 2.1

(a) *$(a, b)\perp_p$, $p\leqq a'\leqq a$, $p\leqq b'\leqq b$ imply $(a', b')\perp_p$.*

(b) *$(a, b)\perp_p$, $(c, a\smile b)\perp_q$, $q\leqq a$ imply $(c\smile a, b)\perp_p$.*

(c) *$(a, b)\perp_p$, $p\leqq c\leqq b$ imply $(c\smile a, b)\perp_c$.*

(d) *$(a_1, a_2, \cdots, a_n)\perp_p$ is equivalent to $(a_i, a_{i+1}\smile\cdots\smile a_n)\perp_p$, $i=1, 2, \cdots, n-1$.*

Proof. Since $(a, b)M$ is equivalent to $(a, b)M^*$ by Theorem 1.4, we can easily prove this theorem by means of techniques similar to those of Wilcox [1].

Now we shall study the relations between the B-covers and independent sets in a lattice L.

Theorem 2.2. *In a lattice L, let $(a_1, a_2, \cdots, a_n)\perp$; then $x=a_{k_1}\smile a_{k_2}\smile\cdots\smile a_{k_t}$ belongs to $B(a_i, a_{i+1}\smile\cdots\smile a_n)$, where k_t is an integer such that $i\leqq k_1<k_2<\cdots<k_t\leqq n$, $i=1, 2, \cdots, n-1$.*

Proof. (1) In case $k_1=i$, since $a_{k_2}\smile a_{k_3}\smile\cdots\smile a_{k_t}\leqq a_{i+1}\smile a_{i+2}\smile\cdots\smile a_n$, we have $(a_i\frown x)\smile((a_{i+1}\smile a_{i+2}\smile\cdots\smile a_n)\frown x)=a_i\smile((a_{i+1}\smile a_{i+2}\smile\cdots\smile a_n)\frown(a_i\smile a_{k_2}\smile\cdots\smile a_{k_t}))=a_i\smile(a_{k_2}\smile\cdots\smile a_{k_t})\smile(a_i\frown(a_{i+1}\smile a_{i+2}\smile\cdots\smile a_n))=a_i\smile(a_{k_2}\smile\cdots\smile a_{k_t})=x$ by $(a_i, a_{i+1}\smile a_{i+2}\smile\cdots\smile a_n)M$ and $a_i\frown(a_{i+1}\smile a_{i+2}\smile\cdots\smile a_n)=0$. Furthermore we have $(a_i\smile x)\frown(a_{i+1}\smile a_{i+2}\smile\cdots\smile a_n\smile x)=x$, since $a_i\leqq x$. Thus x belongs to $B(a_i, a_{i+1}\smile\cdots\smile a_n)$.

(2) In case $k_1>i$, we have $a_i\frown x=0$ since $a_i\frown x\leqq a_i\frown(a_{i+1}\smile\cdots\smile a_n)=0$ by hypothesis. Hence we have $(a_i\frown x)\smile((a_{i+1}\smile\cdots\smile a_n)\frown x)=x$ from $x\leqq a_{i+1}\smile\cdots\smile a_n$.

On the other hand, $(a_i\smile x)\frown(a_{i+1}\smile\cdots\smile a_n\smile x)=a_i\smile a_{k_1}\smile a_{k_2}\smile\cdots\smile a_{k_t})\frown(a_{i+1}\smile\cdots\smile a_n)=x\smile(a_i\frown(a_{i+1}\smile\cdots\smile a_n))=x$ by $(a_i, a_{i+1}\smile\cdots\smile a_n)\perp$.

Corollary 2.1. *Let L be a finite semi-modular lattice. If $(a_1, a_2, \cdots, a_n)\perp$, then $B(a_i, a_1\smile a_2\smile\cdots\smile a_{i-1}\smile a_{i+1}\smile\cdots\smile a_n)$ contains $x=a_{k_1}\smile a_{k_2}\smile\cdots\smile a_{k_t}$, where k_t is an integer such that $1\leqq k_1<k_2<\cdots<k_t\leqq n$, $i=1, 2, \cdots, n$.*

Proof. This is proved from Lemma 2.6 and Theorem 2.2.

Theorem 2.3. *Let $(a_1, a_2, \cdots, a_n)\perp$. Then we have*

(a) $B(a_1, a_2\smile a_3\smile\cdots\smile a_n)\supset B(a_2, a_3\smile\cdots\smile a_n)\supset\cdots\supset B(a_{n-1}, a_n)$ *in any lattice;*

(b) $B(a_1, a_2\smile a_3\smile\cdots\smile a_n)\supset B(a_1, a_2)$, $B(a_1, a_3\smile a_4\smile\cdots\smile a_n)$ *etc. in a finite semi-modular lattice.*

Proof. (a) If we take x from $B(a_2, a_3\smile\cdots\smile a_n)$, then we have $0\leqq x\leqq a_2\smile a_3\smile\cdots\smile a_n$ by $(a_2, a_3\smile\cdots\smile a_n)\perp$ and [4, Lemma 1]. Hence by $(a_1, a_2\smile\cdots\smile a_n)M$ we have $(a_1\smile x)\frown(a_2\smile a_3\smile\cdots\smile a_n)=x\smile(a_1\frown(a_2\smile a_3\smile\cdots\smile a_n))=x$ since $a_1\frown(a_2\smile\cdots\smile a_n)=0$. Furthermore $(a_1\frown x)\smile((a_2\smile a_3\smile\cdots\smile a_n)\frown x)=(a_2\smile a_3\smile\cdots\smile a_n)\frown x=x$ from $a_1\frown x\leqq a_1\frown(a_2\smile\cdots\smile a_n)=0$. Hence $B(a_1, a_2\smile a_3\smile\cdots\smile a_n)$ contains x, that is, $B(a_1, a_2\smile\cdots\smile a_n)\supset B(a_2, a_3\smile a_4\smile\cdots\smile a_n)$. Similarly we can treat the other cases.

(b) If we take x from $B(a_1, a_3\smile a_4\smile\cdots\smile a_n)$, then we have $0\leqq x\leqq a_1\smile a_3\smile a_4\smile\cdots\smile a_n$ by [4, Lemma 1] and $(a_1, a_3\smile\cdots\smile a_n)\perp$. Now by Lemma 2.6 we have $(a_2, a_1\smile a_3\smile\cdots\smile a_n)\perp$, and hence we have $(a_2\smile a_3\smile\cdots\smile a_n, a_1\smile a_3\smile\cdots\smile a_n)M$ by Lemma 2.3.

Hence we have $P\equiv(a_2\smile a_3\smile\cdots\smile a_n\smile x)\frown(a_1\smile a_3\smile\cdots\smile a_n)=x\smile((a_1\smile a_3\smile\cdots\smile a_n)\frown(a_2\smile a_3\smile\cdots\smile a_n))$ from $(a_2\smile a_3\smile\cdots\smile a_n, a_1\smile a_3\smile\cdots\smile a_n)M$. But $(a_1\smile a_3\smile\cdots\smile a_n)\frown(a_2\smile a_3\smile\cdots\smile a_n)=a_3\smile a_4\smile\cdots\smile a_n$ from $(a_2, a_1\smile a_3\smile\cdots\smile a_n)\perp$. Hence $P=x\smile a_3\smile a_4\smile\cdots\smile a_n$.

Accordingly we have $(a_1\smile x)\frown(a_2\smile a_3\smile\cdots\smile a_n\smile x)=(a_1\smile x)\frown(a_2\smile a_3\smile\cdots\smile a_n\smile x)\frown(a_1\smile a_3\smile\cdots\smile a_n)=(a_1\smile x)\frown P=(a_1\smile x)\frown(a_3\smile a_4\smile\cdots$

$\smile a_n \smile x) = x$ by $x \in B(a_1, a_3 \smile a_4 \smile \cdots \smile a_n)$.

On the other hand, we have $x = (a_1 \frown x) \smile ((a_3 \smile a_4 \smile \cdots \smile a_n) \frown x) \leqq (a_1 \frown x) \smile ((a_2 \smile a_3 \smile \cdots \smile a_n) \frown x) \leqq x$, and hence we have $(a_1 \frown x) \smile ((a_2 \smile a_3 \smile \cdots \smile a_n) \frown x) = x$. Hence x belongs to $B(a_1, a_2 \smile a_3 \smile \cdots \smile a_n)$, that is, $B(a_1, a_2 \smile a_3 \smile \cdots \smile a_n) \supset B(a_1, a_3 \smile a_4 \smile \cdots \smile a_n)$. Similarly we can treat the other cases.

Theorem 2.4. *Let* $(a_1, a_2, \cdots, a_n)\perp$. *Then we have*

(a) $J(a_1, a_2 \smile a_3 \smile \cdots \smile a_n) \supset J(a'_1, a'_2 \smile \cdots \smile a'_n)$, $CJ(a_1, a_2 \smile a_3 \smile \cdots \smile a_n) \subset CJ(a'_1, a'_2 \smile \cdots \smile a'_n)$ *in any lattice,*

(b) $B(a_1, a_2 \smile a_3 \smile \cdots \smile a_n) \supset B(a'_1, a'_2 \smile \cdots \smile a'_n)$ *in a semi-modular lattice of finite length, where* $0 \leqq a'_i \leqq a_i$, $i = 1, 2, \cdots, n$.

Proof. (a) If we take x from $J(a'_1, a'_2 \smile \cdots \smile a'_n)$, then we have $x = (a'_1 \frown x) \smile ((a'_2 \smile a'_3 \smile \cdots \smile a'_n) \frown x) \leqq (a_1 \frown x) \smile ((a_2 \smile a_3 \smile \cdots \smile a_n) \frown x) \leqq x$. Hence $(a_1 \frown x) \smile ((a_2 \smile a_3 \smile \cdots \smile a_n) \frown x) = x$, and x belongs to $J(a_1, a_2 \smile a_3 \smile \cdots \smile a_n)$. Thus we have $J(a_1, a_2 \smile a_3 \smile \cdots \smile a_n) \supset J(a'_1, a'_2 \smile \cdots \smile a'_n)$. Dually we have the other relation.

(b) If we take x from $B(a'_1, a'_2 \smile \cdots \smile a'_n)$, then we have $0 \leqq x \leqq a'_1 \smile a'_2 \smile \cdots \smile a'_n$ by [4, Lemma 1] and Lemma 2.1. Since $a_1 \smile x \leqq a_1 \smile a'_2 \smile \cdots \smile a'_n$, $x \smile a_2 \smile a_3 \smile \cdots \smile a_n \leqq a'_1 \smile a_2 \smile a_3 \smile \cdots \smile a_n$, we have $(a_1 \smile x) \frown (a_2 \smile a_3 \smile \cdots \smile a_n \smile x) = (a_1 \smile x) \frown (a_2 \smile a_3 \smile \cdots \smile a_n \smile x) \frown (a_1 \smile a'_2 \smile a'_3 \smile \cdots \smile a'_n) \frown (a'_1 \smile a_2 \smile a_3 \smile \cdots \smile a_n)$.

Now we have $(a_1 \smile a'_2 \smile a'_3 \smile \cdots \smile a'_n, a_2 \smile a_3 \smile \cdots \smile a_n)M$ by Lemma 2.3, and hence $(a_2 \smile a_3 \smile \cdots \smile a_n, a_1 \smile a'_2 \smile \cdots \smile a'_n)M$ by semi-modularity. Hence we have $(a_2 \smile a_3 \smile \cdots \smile a_n \smile x) \frown (a_1 \smile a'_2 \smile a'_3 \smile \cdots \smile a'_n) = x \smile ((a_2 \smile a_3 \smile \cdots \smile a_n) \frown (a_1 \smile a'_2 \smile \cdots \smile a'_n)) = x \smile a'_2 \smile a'_3 \smile \cdots \smile a'_n$ since $(a_2 \smile a_3 \smile \cdots \smile a_n) \frown (a_1 \smile a'_2 \smile \cdots \smile a'_n) = a'_2 \smile \cdots \smile a'_n$ by $(a_1, a_2 \smile a_3 \smile \cdots \smile a_n)\perp$. In the same way we have $(a_1 \smile x) \frown (a'_1 \smile a_2 \smile \cdots \smile a_n) = x \smile a'_1$ since $a_1 \frown (a'_1 \smile a_2 \smile \cdots \smile a_n) = a'_1$ by $(a_2 \smile a_3 \smile \cdots \smile a_n, a_1)\perp$.

Accordingly we have $(a_1 \smile x) \frown (a_2 \smile a_3 \smile \cdots \smile a_n \smile x) = (a_2 \smile a_3 \smile \cdots \smile a_n \smile x) \frown (a_1 \smile a'_2 \smile a'_3 \smile \cdots \smile a'_n) \frown (a_1 \smile x) \frown (a'_1 \smile a_2 \smile \cdots \smile a_n) = (a'_1 \smile x) \frown (x \smile a'_2 \smile a'_3 \smile \cdots \smile a'_n) = x$ by hypothesis. Furthermore, from the proof of (a), we have $(a_1 \frown x) \smile ((a_2 \smile a_3 \smile \cdots \smile a_n) \frown x) = x$. This completes the proof of (b).

Theorem 2.5. *Let* L *be a lattice with* 0, *and if* $CJ(a_i, a_{i+1} \smile \cdots \smile a_n) = L$ *for* $i = 1, 2, \cdots, n-1$, *then* $(a_1, a_2, \cdots, a_n)\perp$.

Proof. (1) If we take x from L, then from $x \in CJ(a_i, a_{i+1} \smile \cdots \smile a_n)$ we have $(a_i \smile x) \frown (a_{i+1} \smile \cdots \smile a_n \smile x) = x$, hence $a_i \frown (a_{i+1} \smile \cdots \smile a_n) \leqq x$ for any $x \in L$. But L contains 0, then $a_i \frown (a_{i+1} \smile \cdots \smile a_n) = 0$.

(2) Take x' such that $x' \leqq a_{i+1} \smile \cdots \smile a_n$ in L, then we have by hypothesis $x' = (a_i \smile x') \frown (a_{i+1} \smile \cdots \smile a_n \smile x') = (a_i \smile x') \frown (a_{i+1} \smile \cdots \smile a_n)$. On the other hand, we have $x' \smile (a_i \frown (a_{i+1} \smile \cdots \smile a_n)) = x'$ from $a_i \frown (a_{i+1} \smile \cdots \smile a_n) = 0$.

Accordingly we have $(a_i, a_{i+1} \smile \cdots \smile a_n)M$. From (1), (2) we have

$(a_1, a_2, \cdots, a_n)\perp$.

Definition. We write $(a, S)\perp$ if $(a, x)\perp$ (i.e. $a\frown x=0$ and $(a, x)M$) for all $x\in S$.

Theorem 2.6. *In a lattice, the following statements are equivalent.*

(a) $(a_1, a_2, a_3)\perp$,

(b) $(a_1, J(a_2, a_3))\perp$, $(a_2, J(a_3, 0))\perp$.

Proof. If $(a_1, a_2, a_3)\perp$, and if we take x from $J(a_2, a_3)$, then we have $a_1\frown x=a_1\frown((a_2\frown x)\smile(a_3\frown x))\leqq a_1\frown x\frown(a_2\smile a_3)=0$, and hence we have $a_1\frown J(a_2, a_3)=0$. Furthermore, since $0\leqq J(a_2, a_3)\leqq a_2\smile a_3$, we have $(a_1, J(a_2, a_3))M$ by Lemma 2.1. Thus we have $(a_1, J(a_2, a_3))\perp$, similarly $(a_2, J(a_3, 0))\perp$.

Conversely if $(a_1, J(a_2, a_3))\perp$, and $(a_2, J(a_3, 0))\perp$, then we have $(a_1, a_2\smile a_3)\perp$ and $(a_2, a_3)\perp$ since $a_2\smile a_3\in J(a_2, a_3)$, $a_3\in J(a_3, 0)$. Thus we have $(a_1, a_2, a_3)\perp$.

Theorem 2.7. *In a lattice L with 0, let S be a subset of L with the greatest element, and if $CJ(a, S)=L$, then we have $(a, S)\perp$.*

Proof. Let m be the greatest element of S, and if we take x from L, then we have $(a\smile x)\frown(m\smile x)=x$ from $x\in CJ(a, S)$. Hence we have $a\frown m\leqq x$ for any $x\in L$. However L contains 0 and hence $a\frown m=0$. Thus we have $a\frown S=0$. If we take x' such that $x'\leqq m$ in L, then from $CJ(a, m)\ni x'$, we have $x'=(a\smile x')\frown(m\smile x')=(a\smile x')\frown m$. On the other hand, since $a\frown m=0$, we have $x'\smile(a\frown m)=x'$.
Hence $(a\smile x')\frown m=x'\smile(a\frown m)$, that is $(a, m)M$. Thus we have $(a, S)M$; this completes the proof.

References

[1] L. R. Wilcox: Modularity in the theory of lattices, Ann. Math., **40** (1939).
[2] G. Birkhoff: Lattice Theory, rev. ed., New York (1948).
[3] L. M. Kelley: The geometry of normed lattice, Duke Math. J., **19** (1952).
[4] Y. Matsushima: On the B-covers in lattices, Proc. Japan Acad., **32** (1956).
[5] Y. Matsushima: The geometry of lattices by B-covers, Proc. Japan Acad., **33** (1957).