112. On Operators of Schaefer Class in the Theory of Singular Integral Equations

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The purpose of the present brief note is to observe the theorems of H. Schaefer [4] under the method of J. W. Calkin [1] and A. F. Ruston [3].

1. Let H be a separable Hilbert space. A bounded linear operator a acting on H will be called an operator of *Schaefer class*, or a σ-transformation in Schaefer's sense, if a satisfies

1) the range R, or $R(a)$, is closed and has finite codimension,

2) the null-space N, or $N(a)=\{\xi;\ \xi a=0\}$, is finite dimensional.

Since $\operatorname{codim} R=\dim H/R=\dim N^*$, where N^* is the null-space of the adjoint a^* of a, 1) is equivalent to assume that the range is closed and N^* has finite dimension.

The typical examples of operators of Schaefer class are

I. if c is completely continuous, then $1-c$ is an operator of Schaefer class by the Riesz theory,

II. if d is regular, i.e. d has the bounded inverse d^{-1}, then d is of Schaefer class.

The set S of all operators of Schaefer class is self-adjoint in the sense that $a\in S$ implies $a^*\in S$ [4, Satz 1].

2. Let B be the B^*-algebra of all bounded linear operators acting on H, and let C be the ideal of B consisting of all completely continuous members. The natural homomorphism of B onto the quotient algebra B/C will be denoted by $\sharp$. The aim of the present note is to show

THEOREM. *An operator a is of Schaefer class if and only if a is regular modulo C,* i.e. *$a^\sharp$ has an inverse in B/C.*

3. The proof of the sufficiency is contained essentially in [4, §2 Hilfssatz]. Let $ab\equiv ba\equiv 1 \mod C$, that is, $ab=1-c$ with $c\in C$ and $ba=1-c'$ with $c'\in C$. By the well-known theory of Riesz, $N(ab)$ and $N^*(ba)$ are finite-dimensional, whence N and N^* are also. Thus it remains to show that Ha is closed. Again, by the Riesz theory, $1-c$ gives one-to-one bicontinuous (isomorphic) mapping of a (closed) subspace F onto another. Therefore, Fa is closed and so Ha is closed too, since F has finite-codimension.

The proof of necessity is same as that of [4, Satz 12]. By the assumption a gives an isomorphism of $N^\perp$ onto R, whence the inverse b' of a on R exists and is bounded by a theorem of Banach. If p is a (finite-dimensional) projection on N, then $b=(1-p)b'$ belongs to B. It is clear by the construction that ab equals to $1-p$. Since p belongs to C, ab is congruent to 1 modulo C. The right-inverse will be obtained using the final remark of §1.

4. Now we are able to deduce the following theorems of Schaefer: (1) *S is an open semigroup in B* [4, Sätze 2 & 4] since the regular elements of B/C form an open set and $\sharp$ is continuous; (2) *S is invariant modulo C*, i.e. $S+C\leqq S$ [4, Satz 5]; (3) *if $a=xy$ belongs to S then either both x and y belong to S or both fail to belong* [4, Satz 12] since $y^\sharp=x^{\sharp-1}a^\sharp$ is regular when $x^{\sharp-1}$ exists; and (4) *if $xay=1$ and $x, y\in S$ then $a\in S$*, because the hypothesis and (3) imply $ay\in S$ which implies by (3) the conclusion. Finally we shall show a principal lemma of Schaefer [4, §2 Hilfssatz] which is applied by him to singular integral equations: *If a^2-b^2 is regular, $ab=ba$, $s^2=1$, $sa\equiv as$* mod C and *$sb\equiv bs$* mod C, *then $a+bs\in S$*. Using the hypothesis we have $(a+bs)(a-bs)\equiv a^2-b^2$ mod C, whence $(a+bs)(a-bs)x\equiv 1$ mod C by some x, that is, $(a+bs)^\sharp$ has a right-inverse. Similarly, $a+bs$ has left-inverse modulo C. This is required.

5. Since S is open, the complement K of S gives a spectrum $\wedge_K(a)$ in the sense of Halmos-Lumer [2], and $\wedge_K(a)$ is a compact set in the complex plane. If a is hermitian then $\wedge_K(a)$ is the condensed spectrum in the sense of Calkin [1].

References

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