111. A Note on Some Topological Spaces

By Shouro KASAHARA
Kobe University

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This short note has two purposes: one of them is to determine a topological space treated in [2], that is, it will be shown below that a Hausdorff space satisfying one of the conditions listed in Theorem 2 of [2] is nothing more than a finite set; and the other is to point out a more fundamental property of weakly compact spaces.*)

Let $\mathfrak{S}=\{O_\alpha\}_{\alpha\in A}$ be a family of subsets of a topological space E. We say that $\mathfrak{S}$ is *point finite* if each point of E belongs at most to a finite number of the members of $\mathfrak{S}$. The family $\mathfrak{S}$ is said to be *locally finite* if each point of E possesses a neighbourhood which intersects at most finitely many members of $\mathfrak{S}$; and $\mathfrak{S}$ is *star finite* if each member of $\mathfrak{S}$ intersects at most finitely many members of $\mathfrak{S}$. Moreover, $\mathfrak{S}$ is termed *weakly locally finite* if $\mathfrak{S}$ is locally finite as a family of subsets of the subspace $\bigcup_{\alpha\in A} O_\alpha$ of E (i.e. if each point of $\bigcup_{\alpha\in A} O_\alpha$ possesses a neighbourhood which intersects finitely many members of $\mathfrak{S}$). If the set A of indices is a finite set, the family $\mathfrak{S}$ is called *finite*. Obviously, a star finite family is weakly locally finite, a locally finite family is weakly locally finite, and a weakly locally finite family is point finite.

THEOREM 1. *The following conditions on a Hausdorff space E are equivalent*:

(1) *Every point finite open covering of E is finite.*

(2) *Every point finite family of open sets of E is finite.*

(3) *Every weakly locally finite family of open sets of E is finite.*

(4) *Every star finite family of open sets of E is finite.*

(5) *Every family of pairwise disjoint open sets of E is finite.*

(6) *E is a finite set.*

Proof. It will suffice to prove that (5) implies (6). To prove this, it is sufficient to show that each point of E is open. Suppose that there exists a point $x\in E$ which is not open. Then, if V_0 is a neighbourhood of x, we can find a point $x_1\in V_0$ distinct from x, and then we can choose disjoint open sets V_1 and O_1 such that $x\in V_1$, $x\in O_1$ and $V_1\subseteq V_0$, $O_1\subseteq V_0$. Thus, by induction, it is easy to construct a sequence $\{V_n\}$ of neighbourhoods of x and a sequence $\{O_n\}$ of open

*) For the definition of weakly compact space (espace faiblement compact), see [3 or 4].

sets such that $V_{n-1} \supseteq V_n$, $V_{n-1} \supseteq O_n$ and $V_n \frown O_n = \phi$ for every positive integer n. But, since the family $\{O_n\}$ consists of pairwise disjoint open sets, this contradicts the condition (5).

In a letter of June 22, 1957, to the author, Prof. S. Mardešić has kindly pointed out an error in [1] that though a characterization of pseudo-compact spaces is cited in [1] as a result of S. Mrówka, the truth is that it is due to J. Kernstan and not to S. Mrówka, and further he has kindly informed that the result obtained by S. Mrówka in this field is as follows: A space is pseudo-compact if and only if every locally finite open covering of it is finite.

The circumstance between these results (see also Theorem 1 of [3]) will become apparent in view of the following

THEOREM 2. *A topological space E is weakly compact if and only if every locally finite family of open sets of E is finite.*

Proof. Since every family of pairwise disjoint open sets is whether locally finite or not, the "only if" part of the theorem is evident. In order to prove the "if" part of the theorem, let us suppose that there exists a locally finite family $\{O_\alpha\}_{\alpha\in A}$ consisting of infinitely many open sets. Then, for any point x_1 in $\bigcup_{\alpha\in A} O_\alpha$, we can find an open neighbourhood U_1 which intersects only a finite number of O_α's. Suppose now that we have obtained pairwise disjoint open sets, $U_1, U_2, \cdots, U_n$ such that each U_i intersects only a finite number of O_α's. Then since the power of A is infinite, there exists a member O_α^{n+1} of the family for which we have $U_{n+1} \frown U_i = \phi$ for $i = 1, 2, \cdots, n$. Let x_{n+1} be a point of O_α^{n+1}, and let $U_{n+1} \subseteq O_\alpha^{n+1}$ be an open neighbourhood of x_{n+1} which intersects finitely many members of $\{O_\alpha\}_{\alpha\in A}$. It is clear that the open set U_{n+1} is disjoint from each U_i, $i=1, 2, \cdots, n$. We have thus a sequence $\{U_n\}$ of pairwise disjoint open sets. But, from the assumption that the space E is weakly compact, it follows that $\{U_n\}$ is not locally finite. Since $U_n \subseteq O_\alpha^n$ for any positive integer n, this is absurd. Therefore the family $\{O_\alpha\}_{\alpha\in A}$ must be finite.

References

[1] K. Iséki and S. Kasahara: On pseudo-compact and countably compact spaces, Proc. Japan Acad., **33**, 100–102 (1957).

[2] S. Kasahara: Boundedness of semicontinuous finite real functions, Proc. Japan Acad., **33**, 183–186 (1957).

[3] S. Kasahara: On weakly compact regular spaces. II, Proc. Japan Acad., **33**, 255–259 (1957).

[4] S. Mardešić et P. Papić: Sur les espaces dont toute transformation réelle continue est bornée, Glasnik Mat.-Fiz. i Astr., **10**, 225–232 (1955).