106. On the Continuity of Norms

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Let R be a universally continuous[1] normed semi-ordered linear space. A norm on R is said to be *continuous*, if $a_\nu \downarrow_{\nu=1}^{\infty} 0$[2] implies $\inf_{\nu=1,2,\cdots} \| a_\nu \| = 0$. The importance of continuity of a norm is in the fact that every norm-bounded linear functional on R is, roughly speaking, represented by a continuous function on the proper space of R (cf. [3]). In this note, we consider some conditions of the continuity of norms on R. We use the terminologies and notations in [4].

H. Nakano obtained the following three conditions of continuity:

Theorem A. *If every norm-bounded linear functional on R is continuous,*[3] *the norm is continuous* [4, Theorem 31.10].

Theorem B. *If a norm on R is separable and semi-continuous,*[4] *it is continuous* [4, Theorem 30.27].

Theorem C. *If a norm on R is uniformly monotone and complete, it is continuous* [4, Theorem 30.22].

In the sequel, the set of a type: $\{x;\ a \leq x \leq b\}$ is called a *segment*.

We know that the semi-continuity implies the completeness of segments [6, Theorem 3.3]. We shall replace semi-continuity of a norm by the completeness of segments of R in proving the continuity of a norm.

A general condition for continuity is contained in

Lemma 1. *A norm on R is continuous, if and only if every segment of R is complete and the norm satisfies the condition:*

(1) $[p_\nu][p_\mu]=0$,[5] $\nu \neq \mu$ $(\nu, \mu=1,2,\cdots)$ *implies* $\lim_{\nu\to\infty} \| [p_\nu]a \| = 0$ $(a \in R)$.

Proof (cf. [3, Satz 14.3]). If the norm is continuous, it is semi-continuous, hence every segment is complete. For $a \in R$ and $[p_\nu][p_\mu]=0$, $\nu \neq \mu$ $(\nu, \mu=1,2,\cdots)$, we have $(o)\text{-}\lim_{\nu\to\infty} [p_\nu]a = 0$,[6] hence by continuity

1) *Universal continuity* means that for any $a_\lambda \geq 0$ $(\lambda \in \Lambda)$ there exists $\bigcap_{\lambda\in\Lambda} a_\lambda$.

2) $a_\nu \downarrow_{\nu=1}^{\infty} a$ means that $a_\nu \geq a_{\nu+1}$ $(\nu=1,2,\cdots)$ and $\bigcap_{\nu=1}^{\infty} a_\nu = a$.

3) A linear functional $\tilde{a}$ on R is said to be *continuous* (resp. *universally continuous*), if for any $a_\nu \downarrow_{\nu=1}^{\infty} 0$ (resp. $a_\lambda \downarrow_{\lambda\in\Lambda} 0$) $\inf_{\nu=1,2,\cdots} |\tilde{a}(a_\nu)| = 0$ (resp. $\inf_{\lambda\in\Lambda} |\tilde{a}(a_\lambda)| = 0$).

4) A norm is said to be *semi-continuous*, if $0 \leq a_\nu \uparrow_{\nu=1}^{\infty} a$ implies $\sup_{\nu=1,2,\cdots} \| a_\nu \| = \| a \|$.

5) $[p]$ is a projection operator to the normal manifold generated by p: $[p]a = \bigcup_{\nu=1}^{\infty} (\nu |p| \frown a)$ for $0 \leq a \in R$.

6) (o)-lim means order-limit.

$\lim_{\nu\to\infty} \| [p_\nu]a \| = 0$. Conversely let every segment of R be complete and the norm satisfy the condition (1). To see continuity, it is sufficient to prove that $[p_\nu] \downarrow_{\nu=1}^{\infty} 0$ implies $\inf_{\nu=1,2,\cdots} \| [p_\nu]a \| = 0$. The condition (1) implies that $\{[p_\nu]a\}_{\nu=1}^{\infty}$ is a Cauchy sequence, because, if it is not so, there exists a subsequence $\{[p_{\nu_\mu}]\}_{\mu=1}^{\infty}$ such that $\| ([p_{\nu_\mu}] - [p_{\nu_{\mu+1}}])a \| \geq \varepsilon > 0$ $(\mu = 1,2,\cdots)$, contradicting the condition (1). The completeness of $\{x; |x| \leq a\}$ and [4, Theorem 30.1] imply $\lim_{\nu\to\infty} \| [p_\nu]a \| = 0$. Q.E.D.

From Lemma 1, we obtain a slightly general form of Theorem B.

Theorem 1. *If every segment of R is complete and separable, the norm is continuous.*

The proof is almost the same as that of Theorem B.

Next we shall weaken the condition of uniform monotoneness in Theorem C. Firstly we recall some definitions for comparison.

A norm on R is said to be *uniformly monotone*, if for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that

(2) $\quad a \frown b = 0,\ \|a\| = 1,\ \|b\| \geq \varepsilon$ implies $\|a+b\| \geq 1+\delta$.

We define a weaker type of monotoneness: a norm on R is said to be *equally monotone*,[7] if there exists $\delta > 0$ such that

(3) $\quad a \frown b = 0,\ \|a\| = \|b\| = 1$ implies $\|a+b\| \geq 1+\delta$.

This definition is equivalent to the following:

(3′) $\quad a \frown b = 0$ implies $\|a+b\| \geq \mathrm{Min}\,\{\|a\|, \|b\|\} + \delta\cdot\mathrm{Max}\,\{\|a\|, \|b\|\}$.

The dual type of uniform monotoneness is uniform flatness. A norm on R is said to be *uniformly flat*, if for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that

(4) $\quad a \frown b = 0,\ \|a\| = \|b\| = 1$ implies $\|a + \xi b\| \leq 1 + \xi\varepsilon$ for $0 \leq \xi \leq \delta$.

As a weaker type we define: a norm on R is said to be *equally flat*, if there exists $\delta > 0$ such that

(5) $\quad a \frown b = 0,\ \|a\| = \|b\| = 1$ implies $\|a+b\| \leq 2-\delta$.

Duality between uniform monotoneness and uniform flatness is known (cf. [3, §15]).

Lemma 2. *Equal monotoneness and equal flatness are of dual type.*

Proof. Though this is a consequence of the theory of *indicatrices* in [3, §16], we give a direct proof. Suppose first that the norm on R is equally monotone. For $\tilde{a}, \tilde{b} \in \tilde{R}''$,[8] $\tilde{a} \frown \tilde{b} = 0$, $\|\tilde{a}\| = \|\tilde{b}\| = 1$ there exists $0 \leq a \in R$ such that $\|a\| = 1$, $\|\tilde{a} + \tilde{b}\| - \varepsilon \leq (\tilde{a} + \tilde{b})(a)$. Since, as

7) Recently Mr. T. Shimogaki obtained a weaker condition: On the norms by uniformly finite modulars, Proc. Japan Acad., **33**, 304–309 (1957). Also Mr. S. Koshi considered equal monotoneness in studying another problem: Modulars on semi-ordered linear spaces (II), Jour. Fac. Sci. Hokkaidô Univ., Ser. I, **13**, 166–200 (1957).

8) $\tilde{R}''$ denotes the Banach's associated space of R.

easily shown, $(\tilde{a}+\tilde{b})(a)=\sup_{\substack{x\frown y=0\\ x+y=a}}(\tilde{a}(x)+\tilde{b}(y))$, there exist $x_0, y_0\in R$ such that $x_0\frown y_0=0$, $x_0+y_0=a$ $\|\tilde{a}+\tilde{b}\|-2\varepsilon\le\tilde{a}(x_0)+\tilde{b}(y_0)$. If $\|x_0\|\le\|y_0\|$, by (3′) $\|x_0\|\le\|x_0+y_0\|-\delta\|y_0\|\le1-\delta/2$, so $\|\tilde{a}+\tilde{b}\|-2\varepsilon\le\tilde{a}(x_0)+\tilde{b}(y_0)\le\|x_0\|+\|y_0\|\le2-\delta/2$. Since ε is arbitrary, the norm on $\tilde{R}''$ is equally flat by definition. Conversely suppose the norm on R to be equally flat. For $\tilde{a},\tilde{b}\in\tilde{R}''$, $\tilde{a}\frown\tilde{b}=0$, $\|\tilde{a}\|=\|\tilde{b}\|=1$ there exist $0\le a, b\in R$ such that $a\frown b=0$, $\|a\|=\|b\|=1$, $\tilde{a}(a)\ge1-\varepsilon$, $\tilde{b}(b)\ge1-\varepsilon$. Since $\|a+b\|\le2-\delta$ by (5), $\|\tilde{a}+\tilde{b}\|\ge(\tilde{a}+\tilde{b})\left(\frac{a+b}{\|a+b\|}\right)\ge\frac{2-2\varepsilon}{2-\delta}$. Since ε is arbitrary, the norm on $\tilde{R}''$ is equally monotone by definition. Q.E.D.

Theorem 2. *If a norm on R is equally monotone and every segment of R is complete, the norm is continuous.*

The proof is similar to that of [3, Satz 14.3].

In the theory of Banach spaces, some conditions on types of norms are known. A norm on R is said to be *strictly convex*, if for any $\varepsilon>0$ and $a, b\in R$, $\|a\|=\|b\|=1$, $\|a-b\|\ge\varepsilon$, there exists $\delta=\delta(\varepsilon, a, b)>0$ such that

$$\|a+b\|\le2-\delta. \tag{6}$$

If δ is independent of b, the norm is said to be *locally uniformly convex*. Further if δ depends only on ε, the norm is said to be *uniformly convex*. The dual type of convexity is evenness (=differentiability of norms). A norm on R is said to be *even*, if for any $\varepsilon>0$ and for $a, b\in R$, $\|a\|=\|b\|=1$ there exists $\delta=\delta(\varepsilon, a, b)>0$ such that

$$\|a+\xi b\|+\|a-\xi b\|\le2+\xi\varepsilon \quad \text{for} \quad 0\le\xi\le\delta. \tag{7}$$

If δ depends only on ε, the norm is said to be *uniformly even*.

Duality between convexity and evenness is studied in [2, 5].

Theorem 3. *If a norm on R is uniformly convex (or uniformly even), it is equally monotone and equally flat.*

Proof. Let the norm on R be uniformly convex. Then it is uniformly monotone [4, Theorem 30.26], consequently equally monotone. As to equal flatness, $a\frown b=0$, $\|a\|=\|b\|=1$ implies $\|a+b\|\le2-\delta$, where $\delta=\delta(1)$ is given in (6), because $\|a-b\|=\|a+b\|\ge1$. Since uniform convexity and uniform evenness are of dual type (cf. [5, §§76–77]), the assertion for the case of uniform evenness follows from Lemma 2. Q.E.D.

We shall consider the relation between convexity and continuity.

Theorem 4. *If a norm on R is locally uniformly convex and every segment of R is complete, then the norm is continuous.*

Proof. To prove the continuity, it is sufficient to show that

$0 \leq a_\nu \uparrow_{\nu=1}^{\infty} a$, $\| a \| = 1$ implies $\lim_{\nu\to\infty} \| a_\nu - a \| = 0$. Imbedding R into $\bar{\tilde{R}}''$ [9] in a natural way, let $a_\nu \uparrow_{\nu=1}^{\infty} b$ *in* $\bar{\tilde{R}}''$ (b may be different from a). Putting $c_\nu = a_\nu + (a-b)$ ($\nu = 1, 2, \cdots$), we obtain $c_\nu \uparrow_{\nu=1}^{\infty} a$ *in* $\bar{\tilde{R}}''$. If we shall prove $\lim_{\nu\to\infty} \| c_\nu - a \| = 0$, then $\{a_\nu\}_{\nu=1}^{\infty}$ is a Cauchy sequence, consequently the completeness of the segment and [4, Theorem 30.1] imply $\lim_{\nu\to\infty} \| a_\nu - a \| = 0$. Suppose, to the contrary, that $\| c_\nu - a \| > \varepsilon > 0$ ($\nu = 1, 2, \cdots$). By the definition of locally uniform convexity, there exists $\delta > 0$ such that

$$x \in R,\ \| a - x \| > \varepsilon,\ 1 - \delta \leq \| x \| \leq 1 \text{ implies } \| a + x \| \leq 2 - \delta.$$

Since the unit sphere of R is dense in that of $\bar{\tilde{R}}''$ by the topology $\sigma(\bar{\tilde{R}}'', \tilde{R}'')$,[10] there exist $a_{\lambda,\nu} \in R$, $\lambda \in \Lambda_\nu$ ($\nu = 1, 2, \cdots$), where Λ_ν is a directed set, such that $\| a_{\lambda,\nu} \| = \| c_\nu \|$, $\lim_{\lambda \in \Lambda_\nu} \tilde{a}(a_{\lambda,\nu}) = \tilde{a}(c_\nu)$ ($\nu = 1, 2, \cdots$) for every $\tilde{a} \in \tilde{R}''$. We can easily see $\underline{\lim}_{\lambda \in \Lambda_\nu} \| a_{\lambda,\nu} - a \| \geq \| c_\nu - a \| > \varepsilon$ and $\underline{\lim}_{\lambda \in \Lambda_\nu} \| a_{\lambda,\nu} + a \| \geq \| c_\nu + a \|$. Since $c_\nu \uparrow_{\nu=1}^{\infty} a$ implies $\lim_{\nu\to\infty} \| c_\nu \| = \| a \| = 1$ by the semi-continuity of the norm on $\bar{\tilde{R}}''$, there exists ν_0 such that $\| a_{\lambda,\nu} \| = \| c_\nu \| \geq 1 - \delta$ ($\nu \geq \nu_0$). For these ν, we have $\underline{\lim}_{\lambda \in \Lambda_\nu} \| a + a_{\lambda,\nu} \| \leq 2 - \delta$, hence $\| a + c_\nu \| \leq 2 - \delta$, contradicting $\lim_{\nu\to\infty} \| a + c_\nu \| = \| 2a \| = 2$. Q.E.D.

In the above theorem we can not replace locally uniform convexity by strict convexity.

On the other hand, by Theorem A and [4, Theorem 28.11] a necessary and sufficient condition for the continuity of a norm is that every segment is compact (or sequentially complete) by the topology $\sigma(R, \tilde{R}'')$.

A bounded linear functional $\tilde{a} \in \tilde{R}''$ is said to be *supported*, if there exists $a \in R$ such that

$$(8) \qquad \| a \| = 1 \quad \text{and} \quad \tilde{a}(a) = \| \tilde{a} \|.$$

If the unit sphere of R is compact by the topology $\sigma(R, \tilde{R}'')$, the norm is continuous and every $\tilde{a} \in \tilde{R}''$ is supported. We now combine this property with a condition of monotoneness. A norm on R is said to be *monotone*, if

$$(9) \qquad 0 \leq a < b \quad \text{implies} \quad \| a \| < \| b \|.$$

Theorem 5. *If a norm on R is monotone and every positive $0 \leq \tilde{a} \in \tilde{R}''$ is supported, the norm is continuous.*

9) $\bar{R}$ denotes the totality of all universally continuous linear functionals on R, and we put $\bar{\tilde{R}}'' = \bar{R} \frown \tilde{R}''$.

10) $\sigma(R, S)$ denotes the weak topology on R defined by all elements of S.

Proof. For $[p_\nu]\downarrow_{\nu=1}^{\infty}0$, considering $[p_\nu]$ as a projection operator $[p_\nu]^{\tilde{R}''}$ on $\tilde{R}''$ (cf. [4, §18]), put $\boldsymbol{P}=\bigcap_{\nu=1}^{\infty}[p_\nu]^{\tilde{R}''}$. For $0\leq\tilde{a}\in\tilde{R}''$, $\boldsymbol{P}\tilde{a}=\tilde{a}$, there exists $0\leq a\in R$, $\|a\|=1$, $\tilde{a}(a)=\|\tilde{a}\|$, because $\tilde{a}$ is supported by assumption. But since $\boldsymbol{P}\tilde{a}(x)=\lim_{\nu\to\infty}\tilde{a}([p_\nu]x)$ for every $x\in R$, $\tilde{a}([p_\kappa]a)=\boldsymbol{P}\tilde{a}([p_\kappa]a)=\tilde{a}(a)$ $(\kappa=1,2,\cdots)$, namely $\tilde{a}([p_\kappa]a)=\|\tilde{a}\|$ $(\kappa=1,2,\cdots)$. If $\tilde{a}\neq 0$, monotoneness of the norm implies $[p_\kappa]a=a$ $(\kappa=1,2,\cdots)$, hence $a=\bigcap_{\kappa=1}^{\infty}[p_\kappa]a=0$, this is a contradiction. So we have $\boldsymbol{P}=0$. Thus for any $[p_\nu]\downarrow_{\nu=1}^{\infty}0$ we have $\lim_{\nu\to\infty}\tilde{x}([p_\nu]x)=0$, namely every $\tilde{x}\in\tilde{R}''$ is continuous. The assertion follows from Theorem A. Q.E.D.

When we consider normed semi-ordered linear spaces, the assumption of *semi-regularity* (that is, separativeness of the topology $\sigma(R,\bar{R}'')$) is natural.

Theorem 6. *Let R be semi-regular. If every* $\tilde{a}\in\tilde{R}''$ *is supported, the norm is continuous.*

Proof. Without loss of generality, we may assume that there exists a complete element[11] $0\leq\bar{a}\in\bar{R}''$, $\|\bar{a}\|=1$. For $0\leq\tilde{b}\in\tilde{R}''$ $\|\tilde{b}\|=2$, $\bar{a}\frown\tilde{b}=0$ (if it exists), there exists by assumption $a\in R$ such that $\|a\|=1$, $(\bar{a}-\tilde{b})(a)=\|\bar{a}-\tilde{b}\|$. It follows that $\bar{a}(a^-)+\tilde{b}(a^+)=0$, because $\bar{a}\frown\tilde{b}=0$ implies $\|\bar{a}+\tilde{b}\|=\|\bar{a}-\tilde{b}\|$ so $(\bar{a}+\tilde{b})(|a|)=(\bar{a}-\tilde{b})(a)$, hence $(\bar{a}+\tilde{b})(|a|)-(\bar{a}-\tilde{b})(a)=2\{\bar{a}(a^-)+\tilde{b}(a^+)\}=0$. Since $\bar{a}$ is complete, we have $a^-=0$, consequently $\tilde{b}(a)=0$. This shows that $\|\bar{a}-\tilde{b}\|=\|\bar{a}\|=1$, contradicting $\|\bar{a}-\tilde{b}\|\geq\|\tilde{b}\|=2$. Thus for any $\tilde{x}\in\tilde{R}''$, we have $[\bar{a}]\tilde{x}=\tilde{x}$, that is, $\tilde{x}$ is continuous. The assertion follows from Theorem A.

Q.E.D.

Now turning our attention to evenness, we obtain

Theorem 7. *Let every segment of R be complete. If there exists* $\delta>0$ *such that*

(10) $\|\,|a|+\delta a\,\|+\|\,|a|-\delta a\,\|\leq 2+\delta$ *for every* $\|a\|\leq 1$,

then the norm is continuous.

Proof. By Lemma 2 and Theorem 2, it is sufficient to prove the equal flatness of the norm on $\tilde{R}''$. For $\tilde{a},\tilde{b}\in\tilde{R}''$, $\tilde{a}\frown\tilde{b}=0$, $\|\tilde{a}\|=\|\tilde{b}\|=1$ and for any $\varepsilon>0$, there exists $a\in R$ such that $\|a\|=1$, $\|\tilde{a}+\tilde{b}\|=\|\tilde{a}-\tilde{b}\|\leq(\tilde{a}-\tilde{b})(a)+\varepsilon$. So we have $(1+\delta)\|\tilde{a}+\tilde{b}\|=\|\tilde{a}+\tilde{b}\|+\delta\|\tilde{a}-\tilde{b}\|$

$$\leq(\tilde{a}+\tilde{b})(|a|)+\delta(\tilde{a}-\tilde{b})(a)+(1+\delta)\varepsilon$$
$$=\tilde{a}(|a|+\delta a)+\tilde{b}(|a|-\delta a)+(1+\delta)\varepsilon$$

11) $\bar{a}\in\bar{R}$ is said to be *complete*, if $|\bar{a}|(|a|)=0$ implies $a=0$.

consequently by (10), $\|\tilde{a}+\tilde{b}\|\leq\frac{2+\delta+(1+\delta)\varepsilon}{1+\delta}$. Since ε is arbitrary, the norm on $\tilde{R}''$ is equally flat. Q.E.D.

The condition of evenness is more convenient than that of convexity, as is seen in the following:

Theorem 8. *If a norm on R is even and every segment of R is complete, the norm is continuous.*

Proof. Suppose the contrary. There exist by Lemma 1 $a\in R$ and $\{[p_\nu]\}_{\nu=1}^{\infty}$ such that $[p_\nu][p_\mu]=0$, $\nu\neq\mu$, $\bigcup_{\nu=1}^{\infty}[p_\nu]a=a$, $\|[p_\nu]a\|\geq\varepsilon>0$ $(\nu,\mu=1,2,\cdots)$ for some $\varepsilon>0$. The subspace: $\{x;\ [x]\leq[a],\ [p_\nu]x=\xi_\nu[p_\nu]a$ $(\nu=1,2,\cdots)$ for some ξ_ν, $\sup_{\nu=1,2,\cdots}|\xi_\nu|<\infty\}$ is, as a normed linear space, isomorphic to (m), the space of all bounded sequences of real numbers with the usual norm, under the correspondence $x\longleftrightarrow(\xi_\nu)$ for the above x and (ξ_ν), because we have $\sup_{\nu=1,2,\cdots}|\xi_\nu|\,\|a\|\geq\|x\|\geq\varepsilon\sup_{\nu=1,2,\cdots}|\xi_\nu|$. But M. M. Day [1] proved that the space (m) admits no equivalent norm which is even. Q.E.D.

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