129. A Relation between Two Realizations of Complete Semi-simplicial Complexes

By Yukihiro KODAMA

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1. Let $S(X)$ be a singular complex of a topological space X. J. B. Giever [2] constructed a polytope $P(X)$ which is a geometric realization of $S(X)$ and whose homotopy groups are isomorphic to those of X. Let K be a complete semi-simplicial (c.s.s.) complex [1]. S. T. Hu [3] constructed a polytope $P(K)$, which is a geometric realization of K. Hu's realization associated with $S(X)$ is homeomorphic to Giever's realization $P(X)$. J. Milnor [6] has defined a geometric realization $|K|$ of K which is different from that used by Giever and Hu. In this note, we shall show that Milnor's realization $|K|$ has the same homotopy type as Giever-Hu's realization $P(K)$.

2. Let K be a c.s.s. complex. The face and degeneracy maps of K are transformations such that

$$F_i : K_q \to K_{q-1}, \quad q>0, \ i=0,1,\cdots,q,$$
$$D_i : K_q \to K_{q+1}, \quad q\geqq 0, \ i=0,1,\cdots,q,$$

where K_q is the set of q-simplexes of K, and satisfy the following commutation rules:

$$\text{(A)}\qquad \begin{gathered} F_iF_j=F_{j-1}F_i, \ D_iD_j=D_{j+1}D_i, \ F_iD_j=D_{j-1}F_i, \ i<j, \\ F_jD_j=F_{j+1}D_j=\mathit{identity}, \quad D_iD_i=D_{i+1}D_i, \\ F_iD_j=D_jF_{i-1}, \quad i>j+1. \end{gathered}$$

We denote by $\varDelta_q=(0,1,\cdots,q)$ the standard q-simplex. $e_i:\varDelta_{q-1}\to\varDelta_q$ and $d_i:\varDelta_q\to\varDelta_{q-1}$ will denote the simplicial mappings defined by

$$e_i(j)=\begin{cases} j, & 0\leqq j<i, \\ j+1, & i\leqq j<q, \end{cases} \qquad d_i(j)=\begin{cases} j, & 0\leqq j\leqq i, \\ j-1, & i<j\leqq q. \end{cases}$$

Form the topological sum $\tilde{K}=\bigcup_q (K_q\times\varDelta_q)$ with the discrete topology on K_q. Consider the following relations:

(i) $\quad (F_is, x)\approx(s, e_ix), \quad s\in K_q, \ x\in\varDelta_{q-1},$

(ii) $\quad (D_is, x)\approx(s, d_ix), \quad s\in K_q, \ x\in\varDelta_{q+1}.$

Milnor's realization $|K|$ is the identification space formed by reducing $\tilde{K}$ by the relations (i) and (ii). The following lemma is proved easily.

Lemma 1. *Giever-Hu's realization $P(K)$ is the identification space formed by reducing $\tilde{K}$ by the relation* (i).

By Lemma 1 there exist natural projections $g:\tilde{K}\to P(K)$ and $f:P(K)\to|K|$.

Lemma 2. *For each 0-cell v of $|K|$, $f^{-1}(v)$ is homeomorphic to a CW-complex Q* [7] *such that*

1) *Q is contractible in itself,*
2) *Q^n is contractible in Q^{n+1} and has only one n-cell for $n=0,1,2,\cdots$, where Q^j is the j-section of Q.*

Proof. There exists a unique vertex $\tilde{v}$ of K such that $g(\tilde{v}\times\Delta_0)=v$. Let $M(v)$ be the subcomplex of K consisting of all simplexes which lie on $\tilde{v}$ [1, p. 508]. Since the face and degeneracy maps F and D satisfy the commutation rules (A), $M(v)$ has only one n-simplex for $n=0,1,\cdots$. Therefore, $M(v)$ is isomorphic to the singular complex T of one point space and for any two 0-cells v and v' of $|K|$ $M(v)$ and $M(v')$ are isomorphic. Since $f^{-1}(v)$ is Giever-Hu's realization of $M(v)$, $f^{-1}(v)$ is homeomorphic to $Q=P(T)$. By [2, Theorem VI] and [7, Theorem 1] Q is contractible in itself. The property 2) of Q is a consequence of [7, (L) in §5].

Lemma 3. *Let x be an interior point of n-cell σ of $|K|$. Then $f^{-1}(x)$ is homeomorphic to the product complex $\overbrace{Q\times Q\times\cdots\times Q}^{(n+1)\text{-fold}}$.*

Proof. There exists a unique non-degenerate n-simplex τ of K such that $fg(\tau\times\Delta_n)=\sigma$. Then $fg\,|\,\tau\times\Delta_n:\tau\times\Delta_n\to\sigma$ is a characteristic map of σ [7, p. 221]. Since x is an interior point of σ, the set $(fg)^{-1}x\frown(\tau\times\Delta_n)$ consists of only one point. Let $(t_0,t_1,\cdots,t_n)$ be the barycentric coordinates of the point $(fg)^{-1}x\frown(\tau\times\Delta_n)$. Let s be an m-cell of $P(K)$ such that $f(s)=\sigma$. Take the m-simplex $\tilde{s}$ of K such that $g(\tilde{s}\times\Delta_m)=s$. Then $\tilde{s}$ can be expressed uniquely as $D_{j_k+i_k}D_{j_k+i_k-1}\cdots D_{j_k}D_{j_{k-1}+i_{k-1}}\cdots D_{j_r+i_r}D_{j_r+i_r-1}\cdots D_{j_r}\cdots D_{j_1+i_1}D_{j_1+i_1-1}\cdots D_{j_1}\tau$, where $m=n+\sum_{r=1}^{k}(i_r+1)$ and $0\leqq j_1<j_1+i_1<j_1+i_1+1<j_2<\cdots<j_r<j_r+i_r<j_r+i_r+1<j_{r+1}<\cdots<j_k+i_k\leqq m$. By making use of the barycentric coordinates, each point $\tilde{y}$ of $\tilde{s}\times\Delta_m$ such that $fg(\tilde{y})=x$ can be represented as follows: $[t_0,\cdots,t_{l_1-1},(\tilde{q}_{i_1};t_{l_1}),t_{l_1+1},\cdots,t_{l_2-1},(\tilde{q}_{i_2};t_{l_2}),\cdots,t_{l_r-1},(q_{i_r};t_{l_k}),t_{l_k+1},\cdots,t_n]$, where $(t_0,\cdots,t_n)$ is the barycentric coordinates of the point $(fg)^{-1}x\frown(\tau\times\Delta_n)$, $l_1=j_1$, $l_r=j_r-\sum_{p=1}^{r-1}i_p$, $r=2,\cdots,k$, and $\tilde{q}_{i_r}$ is a point of the standard i_r-simplex Δ_{i_r}. Let g_p be the characteristic map of the unique p-cell σ_p of Q induced by the identification map g for $p=0,1,\cdots$. Then the point $g(\tilde{y})=y$ of s can be represented as $[t_0,\cdots,t_{l_1-1},(q_{i_1};t_{l_1}),t_{l_1+1},\cdots,t_{l_2-1},(q_{i_2};t_{l_2}),\cdots,(q_{i_r};t_{l_r}),\cdots,(q_{i_n};t_{l_n}),t_{l_n+1},\cdots,t_n]$, where q_{i_r} is the point of σ_{i_r} such that $g_{i_r}(\tilde{q}_{i_r})=q_{i_r}$. If $y\neq y'$ and $f(y)=f(y')=x$ for $y,y'\in s$, it is obvious that $t_j=t'_j$ for $j=0,\cdots,n$ and $q_{i_r}\neq q'_{i_r}$ for some $1\leqq r\leqq k$ in the above representations of y and y'. Put $N_s=s\frown f^{-1}(x)$. Define a transformation $h_s:N_s\to\overbrace{Q\times Q\times\cdots Q}^{(n+1)\text{-fold}}$ by

$$h_s(y)=(\overbrace{\sigma_0,\cdots,\sigma_0}^{l_1\text{-fold}},q_{i_1},\sigma_0,\cdots,\sigma_0,q_{i_{r-1}},\overbrace{\sigma_0,\cdots,\sigma_0}^{(l_r-l_{r-1}-1)\text{-fold}},q_{i_r},\sigma_0,\cdots,\sigma_0,q_{i_n},\overbrace{\sigma_0,\cdots,\sigma_0}^{(n-l_n)\text{-fold}}),$$

where q_{i_r} is the point of σ_{i_r} in the above representation of y. If $\tilde{s}$ is non-degenerate, N_s consists of only one point y and we define $h_s(y)=(\sigma_0,\cdots,\sigma_0)$. Since Q is a countable CW-complex, the product topology of $Q\times\cdots\times Q$ is consistent with its weak topology by an unpublished result due to Dowker (cf. [4, Lemma 8.1, Appendix]). Therefore h_s is a homeomorphism. If s' is a face of s and $f(s')=f(s)=\sigma$, it is not difficult to prove that $h_{s'}=h_s\,|\,N_{s'}$. Moreover, for each cell $\sigma_{j_1}\times\cdots\times\sigma_{j_{n+1}}$ of $Q\times\cdots\times Q$, we can find an $\left(n+\sum_{r=1}^{n+1}j_r\right)$-cell s such that $f(s)=\sigma$ and $h_s(N_s)=\sigma_{j_1}\times\cdots\times\sigma_{j_{n+1}}$. Define the mapping $h:f^{-1}(x)\to Q\times\cdots\times Q$ by $h\,|\,N_s=h_s$. Since $f^{-1}(x)$ is the weak topology about the collection of closed sets $\{N_s\,|\,f(s)=\sigma,\ s\in P(K)\}$, h is a homeomorphism between $f^{-1}(x)$ and $Q\times\cdots\times Q$ by [7, (A) in §5].

The following lemma is a consequence of Lemmas 2 and 3.

Lemma 4. *For each point x of $|K|$, $f^{-1}(x)$ is a countable and contractible CW-complex.*

By Lemma 4 we can make use of a similar argument as the proof of [5, Theorems 1 and 2] to prove the following theorem:

Theorem. *Let M be the 0-section of $P(K)$ and N the 1-section of $|K|$. Then there exists a continuous mapping $\tilde{f}:|K|\to P(K)$ satisfying the following conditions:*

1) $M\subset\tilde{f}(N)$,

2) *$\tilde{f}\,|\,N$ is a homeomorphism and $f\tilde{f}\,|\,N=$identify,*

3) $\tilde{f}f\simeq 1:(P(K),\ \tilde{f}(N))\to(P(K),\ \tilde{f}(N))$*) *and* $f\tilde{f}\simeq 1:(|K|,\ N)\to(|K|,\ N)$.*)

Especially, Milnor's realization $|K|$ has the same homotopy type as Giever-Hu's realization $P(K)$.

Proof. Consider the mapping $e_i:\Delta_{q-1}\to\Delta_q$ and $d_i:\Delta_q\to\Delta_{q-1}$ in the identification relations i) and ii). Let $\tilde{\Delta}_q$ be the third barycentric subdivision of Δ_q, $q=0,1,\cdots$, such that e_i and d_i induce simplicial mappings $\tilde{e}_i:\tilde{\Delta}_{q-1}\to\tilde{\Delta}_q$ and $\tilde{d}_i:\tilde{\Delta}_q\to\tilde{\Delta}_{q-1}$. Form the topological sum $[K]=\smile_q(K_q\times\tilde{\Delta}_q)$ with the discrete topology on K_q. Let P be the identification space formed by reducing $[K]$ by the relation $(F_is,x)\approx(s,\tilde{e}_ix)$, $x\in\tilde{\Delta}_{q-1}$, $s\in K_q$. Let R be the identification space formed by reducing $[K]$ by the relations $(F_is,x)\approx(s,\tilde{e}_ix)$, $x\in\tilde{\Delta}_{q-1}$, $s\in K_q$, and $(D_is,x)\approx(s,\tilde{d}_ix)$, $x\in\tilde{\Delta}_{q+1}$, $s\in K_q$. Then P and R are subdivisions [7, §9] of $P(K)$ and $|K|$. We shall call P and R the third B-subdivisions of $P(K)$ and $|K|$ respectively.

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*) Let (X,A) and (Y,B) be two pairs of topological spaces and let f_0 and f_1 be two continuous mappings of (X,A) to (Y,B) such that $f_0\,|\,A=f_1\,|\,A$. By $f_0\simeq f_1:(X,A)\to(Y,B)$ we mean that there exists a homotopy $H:X\times I\to Y$ such that $H(x,0)=f_0(x)$, $H(x,1)=f_1(x)$, $x\in X$, and $H(a,t)=f_0(a)$, $t\in I$. By 1 we mean the identity mapping.

Similarly, we can construct the n-th B-subdivisions of $P(K)$ and $|K|$ for $n=0,1,\cdots$. Note that the third B-subdivision induces the first and the second B-subdivisions. We shall construct the mapping $\tilde{f}:R\to P$ satisfying the conditions of Theorem. Let σ be a 1-cell of $|K|$. Since each 0-simplex of K is non-degenerate, we can find a unique 1-cell τ of $P(K)$ such that $f|\tau:\tau\to\sigma$ is a homeomorphism. Define $\tilde{f}:N\to P(K)^1$ by $\tilde{f}|\sigma=(f|\tau)^{-1}$, where $P(K)^i$ is the i-section of $P(K)$. Let $\varphi_t:Q\to Q$ be a homotopy, existing by Lemma 2, such that $\varphi_0=$identity, $\varphi_t(Q^n)\subset Q^{n+1}$, $\varphi_t(\sigma_0)=\sigma_0$ and $\varphi_1(Q)=\sigma_0$. Let v be 0-cell of $|K|$ and ψ_v a homeomorphism of $f^{-1}(v)$ to Q. Define $\Phi_t:f^{-1}(|K|^0)\to f^{-1}(|K|^0)$ by $\Phi_t(y)=\psi_v^{-1}\varphi_t\psi_v(y)$, $y\in f^{-1}(v)$, $v\in|K|^0$, where $|K|^i$ is the i-section of $|K|$. Let x be an interior point of a 1-cell σ of $|K|$ and let $\dot{\sigma}=v\smile v'$. Then if σ has only one 0-cell, $v=v'$. By the proof of Lemm 3, each point of $f^{-1}(x)$ is represented by $(q,\tilde{t},q')$, where $0<\tilde{t}<1$, q and q' are points of $f^{-1}(v)$ and $f^{-1}(v')$ respectively. Let us extend the homotopy Φ_t over $f^{-1}(N)$ by putting $\Phi_t(y)=(\psi_v^{-1}\varphi_t\psi_v(q),\ \tilde{t},\ \psi_{v'}^{-1}\varphi_t\psi_{v'}(q'))$ for $y\in f^{-1}(N-|K|^0)$, where $(q,\tilde{t},q')$ is the above representation of the point y. Obviously Φ_t is a homotopy between the identity mapping and the mapping $\tilde{f}f|f^{-1}(N)$ such that $\Phi_t|\tilde{f}(N)=$identity. Moreover, for k-cell τ of $f^{-1}(N)$, $\Phi_t(\tau)\subset f^{-1}(f(\tau))\frown P(K)^{k+1}$. Assume that $\tilde{f}:|k|^{i-1}\to P(K)^{i-1}$, $i>1$, is constructed as follows:

$1)_{i-1}$ $\tilde{f}|N\smile(|K|^{i-1}\frown R^0)$ is a homeomorphism and $f\tilde{f}|N\smile(|K|^{i-1}\frown R^0)$ $=$identity,

$2)_{i-1}$ $f\tilde{f}\simeq 1:(|K|^{i-1},N)\to(|K|^{i-1},N)$ and for each $(i-1)$-cell σ of $|K|$, $f\tilde{f}|\sigma\simeq 1:\sigma\to\sigma$,

$3)_{i-1}$ $\tilde{f}f|f^{-1}(|K|^{i-1})\simeq 1:(f^{-1}(|K|^{i-1}),\ \tilde{f}(N))\to(f^{-1}(|K|^{i-1}),\tilde{f}(N))$ and for each j-cell τ of P in $f^{-1}(|K|^{i-1})$, $\tilde{f}f|\tau\simeq 1:\tau\to f^{-1}(f(\tau))\frown P^{j+1}$, where P^j and R^j are j-sections of P and R. Let σ be an i-cell of $|K|$. Put $[\sigma]=\sigma-\mathrm{St}\,\dot{\sigma}$, where $\mathrm{St}\,A$ is the open star of the set A taken in R. There exists a homeomorphism h_σ of $[\sigma]$ into $g(\tau\times\Delta_i)$, where τ is a unique non-degenerate i-simplex of K. Let us extend $\tilde{f}$ over $[\sigma]$ by defining $\tilde{f}|[\sigma]=h_\sigma$. Take an i-cell μ of R lying on σ such that $\dot{\mu}\frown\dot{\sigma}=\phi$. Therefore, there exists a unique 0-cell v of the second B-subdivision of $|K|$ such that $f^{-1}(\mu)\subset\mathrm{StSt}\,f^{-1}(v)$, where $\mathrm{St}\,B$ is the open star of the set B taken in P. Suppose that the mapping $\tilde{f}$ is extended over the $(j-1)$-section μ^{j-1} of μ, $j\leqq i$, such that $\tilde{f}(\mu^{j-1})\subset\mathrm{StSt}\,f^{-1}(v)\frown f^{-1}(\mu)\frown P^{j-1}$. Let s be a j-cell of μ. Then $\dot{s}\subset\mu^{j-1}$. Since the second B-subdivision of $P(K)$ is a simplicial complex, $f^{-1}(v)$ is a strong deformation retract of $\mathrm{StSt}\,f^{-1}(v)$. Moreover, since $f^{-1}(v)$ is a contractible CW-complex by Lemma 4, we can extend $\tilde{f}$ over s such that

$\tilde{f}(s)\subset \mathrm{StSt}\, f^{-1}(v)\frown f^{-1}(\mu)\frown P^j$. Therefore we have an extension of $\tilde{f}$ over μ^j and by the induction we have an extension of $\tilde{f}$ over $|K|^i$ such that $\tilde{f}(\mu)\subset f^{-1}(\mu)\frown P^i$ for each i-cell μ lying on the i-cell σ of $|K|$. Since $f\tilde{f}(\mu)\subset \mu$ for each j-cell μ of R, $j\leqq i$, it is obvious that the mapping $\tilde{f}$ satisfies $1)_i$ and $2)_i$. We shall prove that $\tilde{f}$ satisfies $3)_i$. Let σ be an i-cell of $|K|$ with the 0-cell v_j, $j=0,\cdots,i$. Each point y of $f^{-1}([\sigma])$ is represented $((q_0;t_0),\cdots,(q_i;t_i))$, where $q_i\in f^{-1}(v_i)$ and $\sum_{j=0}^{i}t_j=1$, $0<t_j<1$. Define $\Phi_t: f^{-1}(x)\to f^{-1}(x)$ by $\Phi_t(y)=((\psi_{v_0}^{-1}\varphi_t\psi_{v_0}(q_0);t_0),\cdots,(\psi_{v_i}^{-1}\varphi_t\psi_{v_i}(q_i);t_i)$ for $y\in f^{-1}[\sigma])$, where $((q_0;t_0),\cdots,(q_i;t_i))$ is the above representation of y. Let μ be an i-cell of R lying on i-cell σ of $|K|$ such that $\dot{\mu}\frown\dot{\sigma}\neq\phi$. Let τ be an m-cell of P such that $f(\tau)=\mu$. There exists a unique 0-cell v of the second B-subdivision of $|K|$ such that $\tilde{f}f(\tau)\smile\tau\subset \mathrm{StSt}\, f^{-1}(v)$. By the same argument as in the construction of the extension of $\tilde{f}$ over $|K|^i$ we can extend the homotopy Φ_t over $f^{-1}(\sigma)$ such that $\Phi_0=$identity, $\Phi_1=\tilde{f}f\,|\,f^{-1}(|K|^i)$ and for each i-simplex τ of P in $f^{-1}(|K|^i)$ $\Phi_t(\tau)\subset f^{-1}(f(\tau))\frown P^{i+1}$. This shows that the mapping $\tilde{f}$ satisfies $3)_i$. Thus we have constructed the mapping $\tilde{f}$ of $|K|$ into $P(K)$ such that $\tilde{f}\,|\,|K|^i$ satisfies the conditions $1)_i$, $2)_i$ and $3)_i$ for each integer i. It is obvious that the mapping $\tilde{f}$ satisfies the conditions 1)–3) of Theorem by [7, (A) and (I) in §5]. This completes the proof.

In the proof of the above theorem, we have proved the following corollary:

Corollary 1. *Let K be a c.s.s. complex such that each i-simplex for $i>1$ is degenerate. Then Milnor's realization $|K|$ is embedded in Giever-Hu's realization $P(K)$ as a strong deformation retract.*

Finally, by our theorem and [6, Theorem 2], we have the following corollary:

Corollary 2. *Let K and K' be countable c.s.s. complexes. Then the four CW-complexes $P(K)\times P(K')$, $P(K\times K')$, $|K|\times|K'|$ and $|K\times K'|$ have the same homotopy type.*

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