

127. A Remark on Pseudo-compact Spaces

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Following Prof. E. Hewitt, a completely regular space is said to be *pseudo-compact*, if every real continuous function on it is bounded. The present writer [4, Corollary 2] proved a theorem closely related to some of results by R. G. Bartle [1] and T. Isiwata [5]. In this Note, we shall show that it is a characterisation of pseudo-compact spaces. Let $\beta(S)$ be the Čech compactification of a completely regular space S , and let $f(x)$ be a bounded continuous function. Let $f^*(x)$ be the unique extension of $f(x)$ on $\beta(S)$. Then we have the following

Theorem. *Let S be a completely regular space, then the following statements are equivalent:*

- (1) S is pseudo-compact.
- (2) For every sequence of bounded continuous functions $f_n(x)$ ($n=1, 2, \dots$) and $f(x)$ such that $f_n(x)$ converges to $f(x)$ pointwisely, $f_n^*(x)$ is convergent to $f^*(x)$ on $\beta(S)$.
- (3) For every sequence of bounded continuous functions $f_n(x)$ such that $f_n(x) \downarrow 0$, we have $f_n^*(x) \rightarrow 0$ on $\beta(S)$.

Proof. (1) \rightarrow (2) follows from Corollary 2 stated before. Since S is dense in $\beta(S)$, (2) implies (3).

To prove (3) \rightarrow (1), it is sufficient to show that the Dini theorem on monotone sequence of bounded continuous functions holds true on S . I. Glicksberg [2], J. Mařík [6] and the present writer [3] have proved that the property is a characterisation of the pseudo-compactness. Let $f_n(x)$ be a sequence of bounded continuous functions such that $f_n(x) \downarrow 0$, then $f_1(x) \geq f_2(x) \geq \dots$ implies $f_1^*(x) \geq f_2^*(x) \geq \dots$ on $\beta(S)$, and by the condition (3), we have $f_n^*(x) \downarrow 0$. Since the Dini theorem holds true on $\beta(S)$, $f_n^*(x)$ is uniformly convergent to 0 on $\beta(S)$. Therefore $f_n(x)$ is uniformly convergent to 0 on S . This shows (3) \rightarrow (1).

References

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