9. On Eigenfunction Expansions of Self-adjoint Ordinary Differential Operators. II

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§3. We introduce the *characteristic matrix* of H by

$$\begin{aligned} M_{11}&=f_a(l)\cdot f_b(l)[f_a(l)-f_b(l)]^{-1}\\ M_{12}&=M_{21}=(1/2)[f_a(l)+f_b(l)][f_a(l)-f_b(l)]^{-1}\\ M_{22}&=[f_a(l)-f_b(l)]^{-1}\end{aligned} \tag{10}$$

where $f_a(l)$, $f_b(l)$ are the characteristic functions of H. By (1), M_{jk} $(j,k=1,2)$ are regular on the upper and the lower half complex planes $(\Im l\neq 0)$ and

$$M_{jk}(\bar{l})=\overline{M_{jk}(l)} \quad (j,k=1,2). \tag{11}$$

For every real number λ, the limits

$$\rho_{jk}(\lambda)=\lim_{\substack{\delta\to+0\\ \delta'\to+0}}\lim_{\varepsilon\to+0}\pi^{-1}\int_{\delta}^{\lambda+\delta'}\Im M_{jk}(\lambda+i\varepsilon)\,d\lambda \tag{12}$$

exist.[1]

As a function of λ, the matrix function $p(\lambda)=(\rho_{jk}(\lambda))$ is continuous on the right and monotone non-decreasing in the sense that, for $\mu<\lambda$, the symmetric matrix $p(\lambda)-p(\mu)$ is positive semi-definite.[2] Hence by the well-known procedure we can construct the matrix set function $p(B)=(\rho_{jk}(B))$ of bounded Borel sets B on the real line corresponding to $p(\lambda)$. $p(B)$ is positive semi-definite, and completely additive on every bounded Borel set. For every $\nu>0$, the residual terms

$$R_{jk}^{(\nu)}(l)=M_{jk}(l)-\int_{-\nu}^{\nu}(\lambda-l)^{-1}\,d\rho_{jk}(\lambda) \tag{13}$$

are regular in the l-plane except for real l such that $l\leqq-\nu$ or $l\geqq\nu$.[2] For the transformation (2) of the system of fundamental solutions, $\rho_{jk}(\lambda)$ are transformed as follows

$$\rho_{jk}(\lambda)=\int_{0}^{\lambda}\sum_{m,n}\beta_{mj}(\lambda)\,\beta_{nk}(\lambda)\,d\tilde{\rho}_{mn}(\lambda).\text{[3]} \tag{14}$$

By (11), (12) and the regularity of $M_{jk}(l)$ for $\Im l\neq 0$, we have for $\lambda'>\lambda$

$$\rho_{jk}(\lambda')-\rho_{jk}(\lambda)=-\lim_{\substack{\mu\to\lambda+0\\ \mu'\to\lambda'+0}}\lim_{\varepsilon\to+0}(2\pi i)^{-1}\int_{C(\mu',\mu,\alpha,\varepsilon)}M_{jk}(l)\,dl \tag{15}$$

where $C(\mu',\mu,\alpha,\varepsilon)$ means the contour consisting of two oriented polygonal lines whose vertices, in order, are $\mu'+i\varepsilon$, $\mu'+i\alpha$, $\mu+i\alpha$, $\mu+i\varepsilon$

1) Cf. Kodaira [3], Theorem 1.3.
2) Cf. Kodaira [3], Theorem 1.3.
3) Cf. Kodaira [3], p. 932.

and $\mu-i\varepsilon$, $\mu-i\alpha$, $\mu'-i\alpha$, $\mu'-i\varepsilon$, respectively, the real number μ', μ, α, ε being subject to the inequalities $\mu'>\mu$, $\alpha>\varepsilon\geqq 0$.

§4. Theorem 2. Let G be the set of points λ on R such that the characteristic function $f_b(l)$ is meromorphic in a neighbourhood of λ. If we put for $\lambda\in R$ and bounded $B\in\mathfrak{B}$ ($\mathfrak{B}$ is the family of Borel sets on R)

$$\rho(\lambda)=\rho_{11}(\lambda)+\rho_{22}(\lambda) \quad \rho(B)=\rho_{11}(B)+\rho_{22}(B) \quad (\geqq 0)$$

and for $\lambda\in G$

$$g_b(\lambda)=f_b(\lambda)[f_b^2(\lambda)+1]^{-1/2} \quad h_b(\lambda)=[f_b^2(\lambda)+1]^{-1/2},$$ [4]

then

$$\left\{\begin{array}{l}\rho_{11}(B)=\displaystyle\int_B g_b^2(\lambda)\,d\rho(\lambda) \quad \rho_{12}(B)=\rho_{21}(B)=\int_B g_b(\lambda)\,h_b(\lambda)\,d\rho(\lambda)\\ \rho_{22}(B)=\displaystyle\int_B h_b^2(\lambda)\,d\rho(\lambda) \quad (g_b^2(\lambda)+h_b^2(\lambda)=1 \quad \text{for} \quad \lambda\in G)\end{array}\right. \tag{16}$$

for a bounded Borel set B contained in G.

Proof. i) Interval of type I.

We assume at first that $f_b(l)$ is regular on a domain D containing a bounded open interval I on R.

We take four real σ, μ, μ', σ' ($\sigma<\mu<\mu'<\sigma'$) belonging to I. Now we take in (13) a ν such that $\nu>|\sigma|$, $|\sigma'|$.

By (10), for the domain $D-R$, we have

$$\left\{\begin{array}{l}M_{11}(l)=f_b^2(l)M_{22}(l)+f_b(l)\\ M_{21}(l)=M_{12}(l)=f_b(l)M_{22}(l)+1/2.\end{array}\right. \tag{17}$$

Here the last terms $f_b(l)$ and $1/2$ are regular on D by the assumption on $f_b(l)$.

By (13), we have for the domain $D-R$

$$\begin{aligned} f_b^2(l)M_{22}(l)&=f_b^2(l)\int_{-\nu}^{\nu}(\lambda-l)^{-1}\,d\rho_{22}(\lambda)+f_b^2(l)R_{22}^{(\nu)}(l)\\ &=\int_{\sigma}^{\sigma'}f_b^2(\lambda)(\lambda-l)^{-1}\,d\rho_{22}(\lambda)+\int_{\sigma}^{\sigma'}[f_b^2(l)-f_b^2(\lambda)](\lambda-l)^{-1}\,d\rho_{22}(\lambda)\\ &\quad+f_b^2(l)\int_{-\nu}^{\sigma}(\lambda-l)^{-1}\,d\rho_{22}(\lambda)+f_b^2(l)\int_{\sigma'}^{\nu}(\lambda-l)^{-1}\,d\rho_{22}(\lambda)+f_b^2(l)R_{22}^{(\nu)}(l)\\ &=\int_{\sigma}^{\sigma'}f_b^2(\lambda)(\lambda-l)^{-1}\,d\rho_{22}(\lambda)+R_{22}(l). \end{aligned} \tag{18}$$

Here $R_{22}(l)$ is regular on $[D-R]\cup(\sigma,\sigma')$ by the assumptions on $f_b(l)$ and ν. For example

$$\int_{\sigma}^{\sigma'}[f_b^2(l)-f_b^2(\lambda)](\lambda-l)^{-1}\,d\rho_{22}(\lambda)$$

is regular on D, since $[f_b^2(l)-f_b^2(\lambda)](\lambda-l)^{-1}$ is regular on $D\times D$ as a function of (l,λ).

We take a contour $C(\mu',\mu,\alpha,\varepsilon)$ as used in (15) for which $\alpha(>0)$

4) If $f_b(\lambda)=\infty$, we put $g_b(\lambda)=1$, $h_b(\lambda)=0$.

is sufficiently small so that the closed contour $C(\mu', \mu, \alpha, 0)$ and its interior are contained in the domain $[D-R]\cup(\sigma, \sigma')$. We write $C(\varepsilon)$ for such contour $C(\mu', \mu, \alpha, \varepsilon)$ when we regard μ', μ, α as fixed and only ε $(\alpha>\varepsilon>0)$ as variable.

From the first formula of (17) and (18), by Cauchy's integral theorem and Fubini's theorem, we have

$$\begin{aligned}\lim_{\varepsilon\to+0}\int_{C(\varepsilon)} M_{11}(l)\,dl &= \lim_{\varepsilon\to+0}\int_{C(\varepsilon)}\Big(\int_{\sigma}^{\sigma'} f_b^2(\lambda)\,(\lambda-l)^{-1}\,d\rho_{22}(\lambda)\Big)\,dl \\ &= \lim_{\varepsilon\to+0}\int_{\sigma}^{\sigma'} f_b^2(\lambda)\Big(\int_{C(\varepsilon)}(\lambda-l)^{-1}\,dl\Big)\,d\rho_{22}(\lambda).\end{aligned} \tag{19}$$

But by Cauchy's integral formula and its modifications in the case when the point λ lies outside or on the contour, we have

$$\lim_{\varepsilon\to+0}\int_{C(\varepsilon)}(\lambda-l)^{-1}\,dl=\begin{cases}-2\pi i & \text{if } \mu'>\lambda>\mu\\ -\pi i & \text{if } \lambda=\mu' \text{ or } \lambda=\mu\\ 0 & \text{if } \lambda>\mu' \text{ or } \lambda<\mu.\end{cases} \tag{20}$$

On the other hand, we have for real λ, ε such that $\sigma<\lambda\leqq\sigma'$ $0<\varepsilon<\alpha$

$$\begin{aligned}&\Big|\int_{C(\varepsilon)}(\lambda-l)^{-1}\,dl\Big|=\Big|-2i\int_{\mu}^{\mu'}\Im[(\lambda-s-i\varepsilon)^{-1}]\,ds\Big| \\ &=2\int_{\mu}^{\mu'}\varepsilon[(\lambda-s)^2+\varepsilon^2]^{-1}\,ds=2\,(\mathrm{Tan}^{-1}(\mu'-\lambda)\varepsilon^{-1}-\mathrm{Tan}^{-1}(\mu-\lambda)\varepsilon^{-1})\leqq 2\pi.\end{aligned}$$

By (20) and (21), we can take the limit with respect to ε in the last term of (19) inside the integral sign with respect to $\rho_{22}(\lambda)$. Therefore

$$\begin{aligned}\lim_{\varepsilon\to+0}\int_{C(\varepsilon)} M_{11}(l)\,dl &= -2\pi i\int_{\mu}^{\mu'} f_b^2(\lambda)\,d\rho_{22}(\lambda) \\ &+\pi i\, f_b^2(\mu')[\rho_{22}(\mu')-\rho_{22}(\mu'-0)]-\pi i\, f_b^2(\mu)[\rho_{22}(\mu)-\rho_{22}(\mu-0)],\end{aligned} \tag{21}$$

since $\int_{\mu}^{\mu'}$ means $\int_{(\mu,\mu']}$.

Hence by (15), considering that $\rho_{22}(\lambda)$ is right continuous, we have for $\lambda', \lambda\in I(\lambda'>\lambda)$

$$\rho_{11}(\lambda')-\rho_{11}(\lambda)=\int_{\lambda}^{\lambda'} f_b^2(\lambda)\,d\rho_{22}(\lambda). \tag{22}$$

In a quite similar way, starting from the second formula of (17), by making use of (13), (15), we get for $\lambda', \lambda\in I(\lambda'>\lambda)$

$$\rho_{21}(\lambda')-\rho_{21}(\lambda)=\rho_{12}(\lambda')-\rho_{12}(\lambda)=\int_{\lambda}^{\lambda'} f_b(\lambda)\,d\rho_{22}(\lambda). \tag{23}$$

From (22), (23), by the well-known procedure we can conclude that

$$\rho_{11}(B)=\int_B f_b^2(\lambda)\,d\rho_{22}(\lambda)\quad \rho_{21}(B)=\rho_{12}(B)=\int_B f_b(\lambda)\,d\rho_{22}(\lambda) \tag{24}$$

for a Borel set B contained in I. From this, considering the definition of $\rho(\lambda)$, $\rho(B)$, $g_b(\lambda)$, $h_b(\lambda)$, we get (16) for a Borel set B contained in I.

ii) Interval of type J.

Now let $f_b(l)$ have a pole at real l_0. If we take the new system of fundamental solutions $\tilde{s}_1(x,l)=s_2(x,l)$, $\tilde{s}_2(x,l)=-s_1(x,l)$, then by (6), (14), we have

$$\begin{cases} \tilde{f}_b(l)=-f_b^{-1}(l) \quad \tilde{\rho}_{11}(\lambda)=\rho_{22}(\lambda) \quad \tilde{\rho}_{22}(\lambda)=\rho_{11}(\lambda) \\ \tilde{\rho}_{12}(\lambda)=\tilde{\rho}_{21}(\lambda)=-\rho_{12}(\lambda)=-\rho_{21}(\lambda). \end{cases} \tag{25}$$

Hence we can find a bounded open interval J on R containing l_0 such that $\tilde{f}_b(l)$ is regular on a domain D' containing J. Then by the same argument as in i), we get (24) where $\rho_{jk}(\lambda)$, $f_b(l)$ are replaced by $\tilde{\rho}_{jk}(\lambda)$, $\tilde{f}_b(l)$, for a Borel set B contained in J. From this, making use of (25) and considering the definitions of $\rho(\lambda)$, $\rho(B)$, $g_b(\lambda)$, $h_b(\lambda)$, we get (16) for a Borel set B contained in J.

iii) Since any bounded Borel set B contained in G can be decomposed into mutually exclusive Borel sets B_i $(i=1,2,\cdots)$ at most countable in number, each of which is contained in a bounded open interval belonging to one of the above two types I and J, we have (16) for any bounded Borel set B contained in G. q. e. d.

§5. We consider Borel-measurable vector functions $\varphi(\lambda)=(\varphi_1(\lambda), \varphi_2(\lambda))$ on R and put

$$||\varphi||^*=\Big(\int_{-\infty}^{+\infty}\sum_{j,k}\varphi_j(\lambda)\overline{\varphi_k(\lambda)}\,d\rho_{jk}(\lambda)\Big)^{1/2}. \tag{26}$$

Since the matrix $p(\lambda)-p(\mu)$ $(\lambda>\mu)$ is always positive semi-definite, we have $+\infty\geqq||\varphi||^*\geqq0$ and $\mathfrak{H}^*=\{\varphi\,|\,||\varphi||^*<+\infty\}$ constitutes a Hilbert space by this norm $||\varphi||^*$ if we identify two $\varphi', \varphi''\in\mathfrak{H}^*$ such that $||\varphi'-\varphi''||^*=0$. We put for $u(x)\in\mathfrak{H}$[5]

$$||u||=\Big(\int_a^b|u(x)|^2\,dx\Big)^{1/2}.$$

Then $\mathfrak{H}$ constitutes a Hilbert space by this norm $||u||$. Now, for every $u\in\mathfrak{H}$, there is a unique $\varphi(\lambda)=(\varphi_1(\lambda), \varphi_2(\lambda))$ such that

$$\Big\|\varphi-\int_{y_1}^{y_2}s(y,\lambda)\,u(y)\,dy\Big\|^*\to0 \quad (y_1\to a+0,\ y_2\to b-0) \tag{27}$$

where $s(x,l)=(s_1(x,l), s_2(x,l))$.[6] If we make the above φ correspond to u, we have a unitary transformation V from $\mathfrak{H}$ onto $\mathfrak{H}^*$ and the inverse transformation V^{-1} is given by

$$(\varphi_1,\varphi_2)\to\int_{-\infty}^{+\infty}\sum_{j,k}s_j(x,\lambda)\,\varphi_k(\lambda)\,d\rho_{jk}(\lambda) \tag{28}$$

where the integral converges in the mean in the L^2-sense.[7] Also $u\in\mathfrak{H}$

5) Cf. §1.
6) Cf. Kodaira [3], Theorem 1.4, p. 928.
7) Cf. Kodaira [3], Theorem 1.4, p. 928.

belongs to the domain of H if and only if $\lambda\cdot\varphi(\lambda)\in\mathfrak{H}^*$ where $\varphi=Vu$, and then

$$VHu=\lambda\cdot\varphi(\lambda). \tag{29}$$

If we denote the spectral measure on R corresponding to H by $\{E_B \mid B\in\mathfrak{B}\}$ where $\mathfrak{B}$ is the family of Borel sets on R, then for any $u\in\mathfrak{H}$

$$VE_Bu=C_B(\lambda)\cdot\varphi(\lambda) \tag{30}$$

where $\varphi=Vu$ and $C_B(\lambda)$ is the characteristic function[8] of the Borel set B.[9]

§6. In this section, we shall state and prove some results which follow from the formulas of §5 by use of Theorem 1 and Theorem 2.

Let G and $\rho(\lambda)$, $g_b(\lambda)$, $h_b(\lambda)$ be defined as in Theorem 2. In the following, we put $g_b(\lambda)=h_b(\lambda)=0$ for $\lambda\in R-G$.

By (30), the unitary transformation V_G, the restriction of V on $E_G(\mathfrak{H})$, has as its range the closed linear submanifold $\mathfrak{H}_G^*$ of $\mathfrak{H}^*$ consisting of $\varphi\in\mathfrak{H}^*$ vanishing outside G. By Theorem 2 and (26), (28), we have

$$\|\varphi\|^*=\Big(\int_G |h_b(\lambda)\varphi_2(\lambda)+g_b(\lambda)\varphi_1(\lambda)|^2\,d\rho(\lambda)\Big)^{1/2} \tag{31}$$

for $\varphi\in\mathfrak{H}_G^*$, and V_G^{-1} is given by

$$\begin{aligned} V_G^{-1}:\ (\varphi_1(\lambda),\varphi_2(\lambda))\to\int_{-\infty}^{+\infty}&[h_b(\lambda)s_2(x,\lambda)+g_b(\lambda)s_1(x,\lambda)]\\ &\times[h_b(\lambda)\varphi_2(\lambda)+g_b(\lambda)\varphi_1(\lambda)]\,d\rho(\lambda)\end{aligned} \tag{32}$$

where the integral converges in the mean in the L^2-sense.

We denote by $\mathfrak{H}_G^{**}$ the set of functions on R vanishing outside G and square integrable with respect to the measure $\rho(B)$ on G and put

$$\|\psi\|^{**}=\Big(\int_G |\psi(\lambda)|^2\,d\rho(\lambda)\Big)^{1/2} \tag{33}$$

for $\psi(\lambda)\in\mathfrak{H}_G^{**}$. Then $\mathfrak{H}_G^{**}$ constitutes a Hilbert space by this norm $\|\psi\|^{**}$.

Now by (31), (33) and the fact that $g_b^2(\lambda)+h_b^2(\lambda)=1$ for $\lambda\in G$, the transformation U from $\mathfrak{H}_G^{**}$ defined by

$$U:\psi(\lambda)\to(g_b(\lambda)\psi(\lambda),\ h_b(\lambda)\psi(\lambda)) \tag{34}$$

is a unitary transformation from $\mathfrak{H}_G^{**}$ onto $\mathfrak{H}_G^*$ and the inverse transformation U^{-1} is given by

$$U^{-1}:(\varphi_1(\lambda),\varphi_2(\lambda))\to h_b(\lambda)\varphi_2(\lambda)+g_b(\lambda)\varphi_1(\lambda). \tag{35}$$

Hence if we put $W=U^{-1}V_G$, then W is a unitary transformation from $E_G(\mathfrak{H})$ onto $\mathfrak{H}_G^{**}$. By (27), (30), (35), for $u\in\mathfrak{H}$, WE_Gu is given by

$$\Big\|WE_Gu-\int_{y_1}^{b}[h_b(\lambda)s_2(y,\lambda)+g_b(\lambda)s_1(y,\lambda)]u(y)\,dy\Big\|^{**}\to0\ \ (y_1\to a+0) \tag{36}$$

8) This should not be confused with the characteristic functions $f_a(\lambda)$, $f_b(\lambda)$ of the operator H.

9) Cf. Kodaira [3], Theorem 1.4, p. 928.

where the integral has its proper sense with respect to its upper limit b, since the function $k_\lambda(x)=h_b(\lambda)s_2(x,\lambda)+g_b(\lambda)s_1(x,\lambda)$ belongs to $\mathfrak{G}_b'$ for each $\lambda\in G$ by Theorem 1 and the definitions of $g_b(\lambda)$, $h_b(\lambda)$. $k_\lambda(x)$ is also a non-trivial solution of $L[u]=\lambda\cdot u$ for each $\lambda\in G$.

By (32), (34), for $\psi\in\mathfrak{H}_G^{**}$, W^{-1} is given by

$$W^{-1}:\psi(\lambda)\rightarrow\int_{-\infty}^{+\infty}[h_b(\lambda)s_2(x,\lambda)+g_b(\lambda)s_1(x,\lambda)]\psi(\lambda)\,d\rho(\lambda) \tag{37}$$

where the integral converges in the mean in the L^2-sense.

By (29) and (35), E_Gu where $u\in\mathfrak{H}$, belongs to the domain of H if and only if $\lambda\cdot\psi(\lambda)\in\mathfrak{H}_G^{**}$ where $\psi=WE_Gu$, and then

$$WHE_Gu=\lambda\cdot\psi(\lambda). \tag{38}$$

Also by (30) and (35), if $\psi=WE_Gu$ where $u\in\mathfrak{H}$, we have for a Borel set B contained in G

$$WE_Bu=C_B(\lambda)\cdot\psi(\lambda) \tag{39}$$

where $C_B(\lambda)$ is the characteristic function of B on R.

Remark 1. We have stated and proved Theorem 2 and the results in §6 for the end point b, but of course similar results can be obtained for the end point a.

Remark 2. From (38) or (39) we see that H has a simple spectrum on G[10] and from (39), (36), (37) we see that $\{k_\lambda(x)\,|\,k_\lambda(x)=h_b(\lambda)s_2(x,\lambda)+g_b(\lambda)s_1(x,\lambda),\ \lambda\in G\}$ is the set of continuous eigenfunctions for $\lambda\in G$ in the sense of Mautner.[11] Theorem 1 states that the continuous eigenfunction $k_\lambda(x)$ for each $\lambda\in G$ belongs to $\mathfrak{G}_b'$. Also $k_\lambda(x)$ is a non-trivial solution of $L[u]=\lambda\cdot u$ for $\lambda\in G$.

References

[1]-[9], listed at the end of part I, Proc. Japan Acad., **33**, 595 (1957).

10) Cf. Stone [5], Chapter VII.
11) Cf. Mautner [4]. Also cf. Bade and Schwartz [1].