# 7. *Ideals in Semirings*

By Kiyoshi ISÉKI
Kobe University

(Comm. by K. KUNUGI, M.J.A., Jan. 13, 1958)

The theory of semiring was first developed by H. S. Vandiver and he has obtained important results of the subjects. Recently, the study of the theory of semiring was made by S. Bourne [1], H. Zassenhaus [2] and the present author [3].

Let $S$ be a semiring (for the definition of a semiring and ideals, see S. Bourne [1]). An ideal $P$ of $S$ is *prime*, if and only if $AB\subset P$ for two ideals $A$, $B$ implies $A\subset P$ or $B\subset P$. In this paper [3], the present author has given some characterisations of prime ideals. First, we shall show some new criteria for prime ideals.

*Theorem 1. If $P$ is an ideal in the semiring $S$, then the following propositions are equivalent:*

(1) *$P$ is prime ideal.*

(2) *If $(a)$, $(b)$ are principal ideals, and $(a)(b)\in P$, then $a\in P$ or $b\in P$.*

(3) *$aSb\subset P$ implies $a\in P$ or $b\in P$.*

(4) *If $R_1, R_2$ are right ideals and $R_1R_2\subset P$, then $R_1\subset P$ or $R_2\subset P$.*

(5) *If $L_1, L_2$ are left ideals and $L_1L_2\subset P$, then $L_1\subset P$ or $L_2\subset P$.*

(6) *If $R$ and $L$ are right and left ideals respectively in $S$ such that $RL\subset P$, then $R\subset P$ or $L\subset P$.*

(7) *If $(a)(b)\subset P$, then $a\in P$ or $B\in P$.**)

(8) *If $(a)_r(b)_r\subset P$, then $a\in P$ or $b\in P$.*

(9) *If $(a)_l(b)_l\subset P$, then $a\in P$ or $b\in P$.*

In [3], we proved the equivalence of propositions (1), (2), (3), (4) and (5). The implications (3)→(4)→(8)→(2), (3)→(5)→(9)→(2), (6)→(7)→(2) and (2)→(3) are trivial. Therefore we shall show (3)→(6). Let $R$ be a right ideal, and $L$ a left ideal. Suppose that $RL\subset P$ and $R$ is not in $P$. Then there is an element $a$ in $R$ such that $a\notin P$. Hence, for each element $b$ of $L$, we have

$$aSb\subseteq RL\subseteq P.$$

Therefore, proposition (3) implies $b\in P$, and this shows that $L\subset P$. This completes the proof of Theorem 1.

We can prove a theorem of completely prime ideal in semiring which is similar to a result of O. Steinfeld [5].

An ideal $P$ is *completely prime* if and only if, $ab\in P$ for $a$, $b$ in $S$ implies $a\in P$ or $b\in P$. Then we have the following

*) $(a)_r$ is the principal right ideal generated by $a$: $\{aS+na\mid(n=1,2,\cdots),\ s\in S\}$, and $(b)_l$ is the principal left ideal generated by $b$.

*Theorem 2. If P is an ideal in a semiring S, then the following propositions are equivalent:*

(1) *P is completely prime.*

(2) *Let L be a left ideal, and R a right ideal, if* $LR \subset P$, *then* $L \subset P$ *or* $R \subset P$.

(3) $(a)_l(b)_r \subset P$ *implies* $a \in P$ *or* $b \in P$.

Proof. To prove (1)→(2), we suppose that (2) fails. Then we can find a right ideal $R$ and a left ideal $L$ such that $LR \subset P$ and $L$, $R$ are not in $P$. Therefore there are two elements $a \in L$, $b \in R$ such that $ab \in P$ and $a, b \notin P$, which contradict (1). Hence (1)→(2).

(2)→(3) is trivial.

To prove (3)→(1), we suppose $ab \in P$ for some $a$, $b$ in $S$. For elements $s$, $s'$, in $S$ and positive integers $m$, $n$ we have

$$\begin{aligned}(sa+ma)(bs'+nb) &= sabs' \\ &\quad + mabs' + nsab + mnab \\ &\in SabS + mabS + mnab \\ &\subset P.\end{aligned}$$

Therefore $(a)_l(b)_r \subset P$. Hence $(a)_l \subset P$ or $(b)_r \subset P$. This shows that $a \in P$ or $b \in P$. The proof is complete.

Next, we shall consider a relation $RL = R \frown L$ for any right ideal $R$ and any left ideal $L$ in the semiring $S$. For the case of rings, such a relation was considered by L. Kovács [4]. In general, the relation $RL \subseteq R \frown L$ is true.

Suppose that $RL = R \frown L$, then for an element $a$ of $S$, we shall consider $(a)_r = \{as + na \mid s \in S, n = 1, 2, \cdots\}$. From the hypothesis, we have

$$(a)_r = (a)_r \frown S = (a)_r \cdot S = (as + na) \cdot S = aS,$$

hence, we have $a \in aS$. By considering of $(a)_l$, $a \in Sa$, and therefore

$$a \in aS \frown Sa = aS^2a.$$

This shows that there is an element $x$ such that $a = axa$, i.e. $S$ is a *regular* semiring.

Conversely, suppose that $S$ is *regular*, then, for an element $a$ of $R \frown L$, we can find an element $x$ such that $a = axa$. On the other hand, $ax \in R$ $a \in L$, therefore $a \in RL$. Hence $R \frown L = RL$, and we have the following

*Theorem 3. A semiring is regular if and only if*

$$R \frown L = RL$$

*is true for any right ideal R and left ideal L.*

Let $P$ be a completely prime ideal in a semiring $S$, then if $aSb \subset P$, then $a \in P$ or $b \in P$, i.e. $P$ is a prime ideal in $S$. For, suppose $a \notin P$ and $b \notin P$, then $ab \notin P$ since $P$ is completely prime. Further, some $P$ is completely prime, $aba \notin P$, which contradicts $aSb \subset P$.

*Theorem 4. A prime ideal P is completely prime, if and only if* $ab \in P$ *implies* $ba \in P$.

Proof. If $P$ is completely prime and $ab \in P$, then $a \in P$ or $b \in P$. Hence $ba \in P$. Conversely, if $ab \in P$ then we have $abS \subseteq P$. By the hypothesis, $aSb \subseteq P$ and we have $a \in P$ or $b \in P$, since $P$ is prime. This shows that $P$ is completely prime.

Following G. Thierrin [6], we shall define a completely semi-prime ideal as follows: An ideal $P$ in the semiring $S$ is *completely semi-prime*, if and only if $a^2 \in P$ implies $a \in P$.

*Proposition. If an ideal $P$ is completely semi-prime, then $ab \in P$ implies $ba \in P$* (G. Thierrin [6]).

For, $ab \in P$ implies $b(ab)a \in P$, and we have $(ba)^2 \in P$. Hence $ba \in P$.

*Theorem 5. A prime ideal $P$ is completely prime, if and only if it is completely semi-prime.*

Proof. It follows from Theorem 4 and the proposition.

*Corollary. In any commutative semiring, the notions of prime ideals, completely prime ideals and completely semi-prime ideals are identical.*

## References

[1] S. Bourne: On multiplicative idempotents of a potent semiring, Proc. Nat. Acad. U. S. A., **42**, 632-638 (1956).
[2] S. Bourne and H. Zassenhaus: On a Wedderburn-Artin, structure theory of a potent semiring, Proc. Nat. Acad. U. S. A., **43**, 613-615 (1957).
[3] K. Iséki: Ideal theory of semiring, Proc. Japan Acad., **32**, 554-559 (1956).
[4] L. Kovács: A note on regular rings, Publ. Math. Debrecen, **4**, 465-468 (1956).
[5] O. Steinfeld: Remark on a paper of N. H. McCoy, Publ. Math. Debrecen, **3**, 171-173 (1953).
[6] G. Thierrin: Sur les idéaux complètement premiers d'un anneau quelconque, Bull, Acad. Royale de Belgiques, **43**, 124-132 (1957).