5. *Two Theorems on Fourier Transform*

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(Comm. by K. KUNUGI, M.J.A., Jan. 13, 1958)

In "Théorie des Distributions" L. Schwartz stated without complete proof that two topological vector spaces $\mathcal{O}_M$ and $\mathcal{O}'_C$ are topologically isomorphic by Fourier transform. Here we shall give a full proof of the theorem.

On the other hand, according to K. Nomizu, a theorem of the same type was proved in C. Chevalley's lecture. The only difference is that Chevalley introduced into $\mathcal{O}'_C$ the bounded-open topology, regarding it as the space $L(\mathfrak{S}, \mathfrak{S})$ of continuous linear operators from $\mathfrak{S}$ to $\mathfrak{S}$. Thus the proof which we shall give concludes that Schwartz's topology in $\mathcal{O}'_C$ and Chevalley's topology coincide.

Next, in 1936, M. Plancherel and G. Polya have proved two main theorems in their paper [1]; one is an extension of the Paley-Wiener's theorem to the multiple integrals of Fourier, and the other a theorem on a simple relation between the spectre of an entire function of exponential type and the order of increase of the function in different directions. The former has been generalized by L. Schwartz [2] to the case with which distributions with compact carriers are concerned.

Similar generalization of the latter theorem of Plancherel-Polya was indicated by Schwartz. But no proof seems to have been published. In this paper, we shall give the formulation and the proof of the theorem.

The author wishes to express his sincere thanks to Professor T. Iwamura for his helpful advices.

1. Schwartz's theorem. By $\mathfrak{S}^0$ we mean the space of rapidly decreasing continuous functions with the topology defined by semi-norms

$$\rho_k(f)=\sup_x (1+r^2)^k\,|f(x)| \qquad (f\in\mathfrak{S}^0)$$

where k is a positive integer.

We shall give two lemmas.

Lemma 1.1. *Let B be a bounded set of $\mathfrak{S}^0$. Then there exist a function $\varphi\in\mathfrak{S}$ and a constant c such that $|f(x)|\leqq c\,|\varphi(x)|$ for all $f\in B$.*

The proof is classical and so omitted.

Lemma 1.2. *Let $P(x)$ be a polynomial on R^n. Then PT belongs to $\mathcal{O}'_C$ when $T\in\mathcal{O}'_C$, and the mapping $T\in\mathcal{O}'_C\to PT\in\mathcal{O}'_C$ is continuous, according to Schwartz's topology.*

Proof. Let the degree of $P(x)$ be m. It is obvious that $P\varphi(1+r^2)^{-m}\in\mathcal{D}_{L^1}$ for each $\varphi\in\mathcal{D}_{L^1}$. So we have $PT\in\mathcal{O}'_C$, since

$$(1+r^2)^k PT\cdot\varphi=(1+r^2)^{k+m}T\cdot P\varphi(1+r^2)^{-m}.$$

The continuity of the mapping is also obtained from the fact that the mapping $\varphi \in \mathcal{D}_{L^1} \to P\varphi(1+r^2)^{-m} \in \mathcal{D}_{L^1}$ is continuous.

Theorem 1. *$\mathcal{O}_M$ is topologically isomorphic to $\mathcal{O}'_C$ by Fourier transform $\mathcal{F}$* (cf. Note 1, p. 27).

Proof. Since it has been proved in TD (cf. Note 2, p. 27) that $\mathcal{O}_M$ is algebraically isomorphic to $\mathcal{O}'_C$ by Fourier transform $\mathcal{F}$.

First, we shall prove the continuity of the mapping $\mathcal{F}: \mathcal{O}_M \to \mathcal{O}'_C$. Take a neighborhood of 0 in $\mathcal{O}'_C$, $U(B, k)=\{T \in \mathcal{O}'_C;\ |(1+r^2)^k T \cdot \varphi| \leqq 1$ for all $\varphi \in B\}$, where B is a bounded set of $\mathcal{D}_{L^1}$. Suppose $g \in \mathcal{D}_{L^1}$. Then $\mathcal{F}[g]$ belongs to $\mathfrak{S}^0$, since $(1+r^2)^k \mathcal{F}[g]=\mathcal{F}[D_k g]$ is a bounded continuous function, where $D_k \delta = \mathcal{F}[(1+r^2)^k]$ so that $D_k=(1-\Delta)^k$. And the mapping $\mathcal{F}: \mathcal{D}_{L^1} \to \mathfrak{S}^0$ is continuous since $\mathcal{F}$ maps L^1 continuously into L^∞. Therefore $\mathcal{F}[B]$ is a bounded set of $\mathfrak{S}^0$. Thus, by Lemma 2, there exist a function $\varphi \in \mathfrak{S}$ and a constant c such that

$$\sup_{g \in B} |\mathcal{F}[g](x)| \leqq c|\varphi(x)|.$$

Now we set $V(\psi, k)=\{h \in \mathcal{O}_M;\ \sup_x |\psi D_k h(x)| \leqq 1\}$, where

$$\psi=\left[c\int(1+r^2)^{-n}dx\right]^{-1}(1+r^2)^n\varphi.$$

It is clear that $\psi \in \mathfrak{S}$ and therefore $V(\psi, k)$ is a neighborhood of 0 in $\mathcal{O}_M$. We shall show that $\mathcal{F}[V(\psi, k)] \subset U(B, k)$. Suppose $h \in V(\psi, k)$. Then from the equality

$$(1+r^2)^k \mathcal{F}[h] \cdot g = h \cdot \mathcal{F}[(1+r^2)^k g] = h \cdot D_k \mathcal{F}[g]$$

it follows that

$$\begin{aligned} |(1+r^2)^k \mathcal{F}[h] \cdot g| &= |D_k h \cdot \mathcal{F}[g]| \\ &= \left|\int\int\cdots\int D_k h(x) \mathcal{F}[g](x)\,dx\right| \\ &\leqq \int\int\cdots\int |D_k h(x)| \cdot c\,|\varphi(x)|\,dx \\ &= \int\int\cdots\int c(1+r^2)^n |\varphi D_k h(x)| (1+r^2)^{-n}\,dx \leqq 1. \end{aligned}$$

Thus we have $\mathcal{F}[V(\psi, k)] \subset U(B, k)$ which proves the continuity of the mapping $\mathcal{F}: \mathcal{O}_M \to \mathcal{O}'_C$.

Now we shall prove the continuity of the mapping $\mathcal{F}: \mathcal{O}'_C \to \mathcal{O}_M$. We shall show that

$$\{T \in \mathcal{O}'_C;\ \sup_x |\varphi D \mathcal{F}[T](x)| \leqq 1\}$$

is a neighborhood of 0 in $\mathcal{O}'_C$ where $\varphi \in \mathfrak{S}$, and D is a differential operator. Since $D\mathcal{F}[T]=\mathcal{F}[P_D T]$, where P_D is the polynominal $\mathcal{F}[D\delta]$ and since the mapping $T \in \mathcal{O}'_C \to P_D T \in \mathcal{O}'_C$ is continuous by Lemma 2, it is sufficient to show that

$$U_\varphi=\{T \in \mathcal{O}'_C;\ \sup_x |\varphi \mathcal{F}[T](x)| \leqq 1\} \quad (\varphi \in \mathfrak{S})$$

is a neighborhood of 0 in $\mathcal{O}'_C$. Since $\mathcal{F}$ maps $\mathfrak{S}$ onto $\mathfrak{S}$, there exists a function $\psi \in \mathfrak{S}$ such that $\mathcal{F}[\psi]=\varphi$. Then we have

$$\mathcal{F}[T]\varphi(x)=\mathcal{F}[T]\mathcal{F}[\psi](x)=\mathcal{F}[T*\psi](x).$$

We remark that the mapping $T\in\mathcal{O}'_C\to T*\psi\in\mathfrak{S}$ is continuous where $\psi\in\mathfrak{S}$ (§5, VI in TD), and that $\mathcal{F}$ gives a homeomorphism between $\mathfrak{S}$ and $\mathfrak{S}$. From the above equality and this remark, we see that U_φ is a neighborhood of 0 in $\mathcal{O}'_C$. Thus we have completed the proof.

2. Plancherel-Polya's theorem. To formulate the theorem we shall introduce some necessary definitions following [1].

By a direction in R^n we mean a vector $\lambda=(\lambda_1,\cdots,\lambda_n)\in R^n$ satisfying the condition

$$(1)\qquad \sum_{i=1}^{n}\lambda_i^2=1.$$

Let $F(z)$ be an entire function of exponential type. Then there exist positive numbers A and a satisfying

$$(2)\qquad |f(z_1,\cdots,z_n)|<A\exp a(|z_1|+\cdots+|z_n|)$$

for all $z\in C^n$ (complex n-dimentional spaces).

Let $\lambda=(\lambda_1,\cdots,\lambda_n)$ be a direction and $\alpha_1,\cdots,\alpha_n$ be arbitrary real numbers. It follows easily from (1) and (2),

$$\overline{\lim_{r\to+\infty}}\, r^{-1}\log|f(\alpha-i\lambda r)|\leqq na$$

where $\alpha=(\alpha_1,\cdots,\alpha_n)$.

Hence we can define a function of direction

$$(3)\qquad h(\lambda)=\sup_{\alpha}\overline{\lim_{r\to+\infty}}\, r^{-1}\log|f(\alpha-i\lambda r)|.$$

If F belongs to $\mathrm{Exp}\,\mathcal{O}_M$ then, as is well known, $\mathcal{F}[F]=T$ is a distribution belonging to $\mathcal{E}'$. Let K be the spectre of F, that is the carrier of T which is necessarily a compact set in R^n. We introduce another function $\chi(\lambda)$ of direction λ in R^n, associated with K, by

$$(4)\qquad \chi(\lambda)=\operatorname*{Max}_{y\in K}\langle\lambda,y\rangle=\operatorname*{Max}_{y\in K}(\lambda_1y_1+\cdots+\lambda_ny_n).$$

We shall use the following lemma which has been established by Plancherel-Polya.

Lemma 2.1. *If*

1) $f(z_1,\cdots,z_n)$ *is an entire function of exponential type on* C^n,

2) $\lambda=(\lambda_1,\cdots,\lambda_n)$ *is an arbitrary but fixed direction in* R^n,

3) $f(x_1,\cdots,x_n)\in L^1(R^n)$,

then $\mathcal{F}[f](y)=0$

in $H=\{y\in R^n;\ \langle\lambda,y\rangle>h(\lambda)\}$.

Our generalized theorem, of which the proof is needed, is:

Theorem 2. *Let* $h(\lambda)$ *and* $\chi(\lambda)$ *be the functions defined by* (3) *and* (4), *respectively, for* $F\in\mathrm{Exp}\,\mathcal{O}_M$. *Then we have* $h(\lambda)=\chi(\lambda)$.

Remark 1. For $n=1$ the above theorem is a more detailed result than the generalized Paley-Wiener's theorem in [2].

Remark 2. The topology of $\boldsymbol{D}=\mathcal{F}[\mathcal{D}]$ is defined, following [3], as an inductive limit of $\boldsymbol{D}_k=\mathcal{F}[\mathcal{D}_k]$, each of which is topologized by means of a fundamental system of neighborhoods of 0, given by

$$U_{P,a}=\{\Phi\in \boldsymbol{D}_k;\ \sup_{z\in C_a}|P(z)\Phi(z)|\leqq 1\},$$

corresponding to all polynominals P on C^n and all positive numbers a, where $\qquad C_a=\{z=(z_1,\cdots,z_n)\in C^n;\ |\mathcal{J}(z_i)|\leqq a,\ i=1,\cdots,n\}$.

We define the topology of $\operatorname{Exp}\mathcal{O}_M$ by the fundamental system of neighborhoods of 0

$$W(C,V)=\{F\in \operatorname{Exp}\mathcal{O}_M;\ \Phi F\in V \quad \text{for all } \Phi\in C\},$$

corresponding to all compact sets C in $\boldsymbol{D}$ and all open sets V in $\boldsymbol{D}$. $\operatorname{Exp}\mathcal{O}_M$, with the compact-open topology mentioned above is topologically isomorphic to $\mathcal{E}'$ by the transformation $\mathcal{F}$.

Proof of the theorem. First, we prove the inequality $\chi(\lambda)\geqq h(\lambda)$.

Let β_j be a sequence of functions $\in\mathcal{D}$, convergent to Dirac measure δ in $\mathcal{E}'$ and satisfying $\beta_j=\check{\beta}_j$. It is obvious that $\beta_j*T\in\mathcal{E}$ and

$$\overline{\mathcal{F}}[\beta_j*T]=\overline{\mathcal{F}}[\beta_j]F\in \operatorname{Exp}\mathcal{O}_M,$$

where we set $F=\overline{\mathcal{F}}[T]$. Since $\beta_j*T\rightarrow T$ in $\mathcal{E}'$, we have $\overline{\mathcal{F}}[\beta_j*T]\rightarrow F$ in $\operatorname{Exp}\mathcal{O}_M$, by Remark 2.

The topology of $\operatorname{Exp}\mathcal{O}_M$ introduced above being stronger than the topology of point-wise convergence, we conclude

$$\overline{\mathcal{F}}[\beta_j*T](z)\rightarrow F(z) \qquad (j\rightarrow\infty)$$

for each $z\in C^n$. Put $\chi_j(\lambda)=\operatorname*{Max}_{y\in K_j}\langle\lambda,y\rangle$ where K_j denotes the carrier of β_j*T. Let us show

$$\lim_{j\rightarrow\infty}\chi_j(\lambda)=\chi(\lambda).$$

Let ε be any positive number. Since $|\langle\lambda,x\rangle-\langle\lambda,y\rangle|\leqq|x-y|$ and since the ε-neighborhood of Carr. (T) contains Carr. $(T*\beta_j)$ for all sufficiently large numbers j, we have $\chi_j(\lambda)\leqq\chi(\lambda)+\varepsilon$ for such j.
(It is sufficient to have this inequality for the proof of our theorem.) Now choose a point $y\in$ Carr. (T) so as to satisfy $\chi(\lambda)=\langle\lambda,y\rangle$. Then $\langle T,\alpha\rangle\neq 0$ for some $\alpha\in\mathcal{D}$ with Carr. (α) contained in the ε-neighborhood W of y. $\langle T*\beta_j,\alpha\rangle=\langle T,\beta_j*\alpha\rangle$, convergent to $\langle T,\alpha\rangle$, is different from 0 for all sufficiently large j and, for such j, the neighborhood W contains a point of Carr. $(T*\beta_j)$, which implies

$$\chi_j(\lambda)\geqq\chi(\lambda)-\varepsilon.$$

Thus we have

$$\lim_{j\rightarrow\infty}\chi_j(\lambda)=\chi(\lambda).$$

From the definition of Fourier transform, it follows

$$\mathcal{F}[\beta_j]F(\alpha-i\lambda r)=\int\int\cdots\int\exp\left(\langle\lambda,y\rangle r+i\langle\alpha,y\rangle\right)T*\beta_j(y)\,dy.$$

Here we consider a family of functions $\{E_{\lambda,\alpha,r}(y)\}$ defined by

$$E_{\lambda,\alpha,r}(y)=\exp\left(\langle\lambda,y\rangle r+i\langle\lambda,\alpha\rangle\right).$$

When such a function of y is subjected to a differential operator

D with constant coefficients, we have

(5) $$DE_{\lambda,\alpha,r}(y)=P(\lambda,\alpha,r)E_{\lambda,\alpha,r}(y)$$

where $P(\lambda,\alpha,r)$ denotes a polynomial of variables λ,α and r.

Now let Ω and Ω_1 be $\frac{\varepsilon}{2}$- and $\frac{\varepsilon}{3}$-neighborhoods of Carr. (T) and $\rho(x)$ be a function belonging to $\mathcal{D}$, which is equal to 1 on Ω_1 and 0 for $x\notin\Omega$. Then

$$\chi(\lambda)-\langle\lambda,y\rangle+\frac{\varepsilon}{2}>0 \quad \text{for all } y\in\Omega,$$

and with the aid of (5) we can conclude that

$$\rho(y)E_r(y)=\rho(y)[\exp(\chi(\lambda)+\varepsilon)r]^{-1}E_{\lambda,\alpha,r}(y)$$

forms a bounded set in $\mathcal{E}_y$ when λ,α are fixed and $r>0$ varies.

For all sufficiently large j, we have Carr. $(T*\beta_j)\subset\Omega_1$ and so

$$T*\beta_j\cdot E_r=T*\beta_j\cdot\rho E_r\rightarrow T\cdot\rho E_r=T\cdot E_r$$

uniformly for all $r>0$ when $j\rightarrow\infty$, since $T*\beta_j\rightarrow T$ in $\mathcal{E}'$. Hence there exists a constant $M(\lambda,\alpha)>0$, depending only on λ and α, such that

$$|T*B_j\cdot E_r|\leqq M(\lambda,\alpha)$$

for all $r>0$ and for all sufficiently large j, which implies

$$r^{-1}\log|\mathcal{F}[\beta_j]F(\alpha-i\lambda r)|\leqq\chi(\lambda)+\varepsilon+r^{-1}\log M(\alpha,\lambda).$$

Let $j\rightarrow\infty$ and then $r\rightarrow\infty$; we thus obtain

$$\overline{\lim_{r\rightarrow\infty}}\, r^{-1}\log|F(\alpha-i\lambda r)|\leqq\chi(\lambda)+\varepsilon$$

for all ε, which proves $h(\lambda)\leqq\chi(\lambda)$.

Now take $\beta\in\mathcal{D}$ with Carr. $(\beta)\subset S_\varepsilon=\{x;|x|<\varepsilon\}$. Then we have, from the classical Paley-Wiener's theorem,

$$\overline{\lim_{r\rightarrow\infty}}\, r^{-1}\log|\mathcal{F}[\beta](\alpha-i\lambda r)F(\alpha-i\lambda r)|$$

$$\leqq\overline{\lim_{r\rightarrow\infty}}\, r^{-1}\log|\mathcal{F}[\beta](\alpha-i\lambda r)|+\overline{\lim_{r\rightarrow\infty}}\, r^{-1}\log|F(\alpha-i\lambda r)|\leqq\varepsilon+h(\lambda).$$

Setting $h_\beta(\lambda)=\sup_\alpha\overline{\lim_{r\rightarrow\infty}}\log|\mathcal{F}[\beta]F(\alpha-i\lambda r)|$, we have

$$h_\beta(\lambda)\leqq\varepsilon+h(\lambda).$$

Now we remark that $\mathcal{F}[\beta]F\in L^1(R^n)$. Thus by the above-mentioned lemma, it holds

$$\mathcal{F}[\overline{\mathcal{F}}[\beta]F](y)=\beta*T(y)=0$$

for all y satisfying $h_\beta(\lambda)<\langle\lambda,y\rangle$ (and accordingly for all y satisfying $h(\lambda)+\varepsilon<\langle\lambda,y\rangle$).

Consequently we have

$$U_\varepsilon=\{y;\langle\lambda,y\rangle\leqq h(\lambda)+\varepsilon\}\supset\text{Carr.}(T*\beta).$$

Since Carr. (β) can be taken in any neighborhood of the origin,

$$U_\varepsilon\supset\text{Carr.}(T)$$

for every $\varepsilon>0$, which means that

$$\{y;\langle\lambda,y\rangle\leqq h(\lambda)\}\supset\text{Carr.}(T).$$

Thus we obtain the inequality $\chi(\lambda) \leqq h(\lambda)$ which completes the proof of $\chi(\lambda)=h(\lambda)$.

Note 1. By the Fourier transform $\mathcal{F}[f]$ of a function $f \in \mathfrak{S}$ we mean the function

$$\mathcal{F}[f](y)=\int\int\cdots\int f(x)\exp(-i\langle x, y\rangle)\,dx.$$

$\mathcal{F}[T]$, the transform of the distribution $T \in \mathfrak{S}'$, is defined as usual by

$$\langle \mathcal{F}[T], f\rangle = \langle T, \mathcal{F}[f]\rangle,$$

where $f \in \mathfrak{S}$.

Note 2. We abbreviate "Théorie des Distributions" to TD.

References

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