# 4. A Note on the Integration by the Method of Ranked Spaces

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§ 1. Prof. K. Kunugi showed in his note "Application de la méthode des espaces rangés à la théorie de l'intégration. I"[1] that a new integration can be constructed by the method of ranked spaces,[2] and suggested that the development of his theory could be generalized for functions on abstract spaces — for example, locally compact topological groups. In this note, we shall consider the locally compact group $G$ and we shall show that the construction of integrals can be done without changing any detail of the preceding note.

Let $G$ be a locally compact group, $m$ be a Haar measure in $G$,[3] that is, a Borel measure in $G$, such that $m(U)>0$ for every non empty Borel open set $U$, and $m(xE)=m(E)$ for every Borel set $E$, and for every element $x$ of $G$.

First we shall remark that, in a locally compact group there is a fundamental system of neighbourhoods of unit element $e$, which consists of neighbourhoods whose boundaries are of measure zero.

Let $V$ be a compact neighbourhood of unit element $e$ whose boundary is of measure zero, and from now on our considerations are restricted to the fixed $V$.

Let the family $\mathcal{O}$ be a totality of open sets in $V$ whose boundaries are of measure zero. Then,

(1) If $O_1 \in \mathcal{O}$, $O_2 \in \mathcal{O}$ then $O_1 \cup O_2 \in \mathcal{O}$, $O_1 \cap O_2 \in \mathcal{O}$.

(2) If $O_1 \in \mathcal{O}$, $O_2 \in \mathcal{O}$ then $O_1 \cap (V-\overline{O}_2) \in \mathcal{O}$.

The vector space over the field of real numbers generated by characteristic functions of sets in $\mathcal{O}$ is denoted by $\Phi$. To $f \in \Phi$ correspond a finite number of disjoint sets $O_i \in \mathcal{O}$ $(i=1, 2, \cdots, n)$ and

$$f(x)=\sum_{i=1}^{n} \alpha_i \chi_{O_i}(x)$$

where $\chi_{O_i}$ is a characteristic function of $O_i$, and $\alpha_i$ is a real number. Two functions of $\Phi$, $f(x)$, $g(x)$ are identified when they are different only on the boundary of $O \in \mathcal{O}$. Obviously if $f \in \Phi$, $g \in \Phi$ then $f+g \in \Phi$,

1) K. Kunugi: Application de la méthode des espaces rangés à la théorie de l'intégration. I, Proc. Japan Acad., **32**, 215-220 (1956).

2) K. Kunugi: Sur les espaces complets et régulièrement complets. I, II, Proc. Japan Acad., **30**, 553-556, 912-916 (1954).

3) On Haar measure, see for example P. R. Halmos; Measure Theory, New York (1950).

$\alpha f \in \Phi$ ($\alpha$ is real), and $|f| \in \Phi$, the set of point of discontinuity of $f$ is of measure zero. For $f \in \Phi$ we define its integral

$$\int f(x)\,dx = \sum_{i=1}^{n} \alpha_i m(O_i)$$

where $f(x) = \sum_{i=1}^{n} \alpha_i \chi_{O_i}(x)$. This integral is clearly linear (with respect to $f$) and $\int |f(x)|\,dx = 0$ implies $f(x)=0$. If $f(x) \geqq 0$ then $\int f(x)\,dx \geqq 0$, finally

$$\left|\int f(x)\,dx\right| \leqq \int |f(x)|\,dx \leqq \sup_{x\in V} |f(x)| \cdot \sum_{i=1}^{n} m(O_i).$$

By the well-known development of integral theory we proceed to enlarge the class of integrable functions and its integrals.[4] First we can prove following two important lemmas:

For every sequence $\{f_n(x)\}$ ($f_n \in \Phi$, $n=1,2,\cdots$) which decreases to zero almost everywhere, the sequence of values of their integrals also tends to zero.

If for an increasing sequence $\{f_n(x)\}$ ($f_n \in \Phi$, $n=1,2,\cdots$) the values of their integrals have a common bound, then the sequence $\{f_n(x)\}$ tends almost everywhere to a finite limit.

In this situation, we set $\Phi_1$ the class of limit functions of increasing sequence $\{f_n(x)\}$ ($f_n \in \Phi$, $n=1,2,\cdots$) having a common bound of their integrals. If $f \in \Phi_1$ and almost everywhere $\lim_{n\to\infty} f_n(x) = f(x)$, where $\{f_n\}$ is defining sequence of $f$, we define the integral of $f(x)$:

$$\int f(x)\,dx = \lim_{n\to\infty} \int f_n(x)\,dx.$$

This integral does not depend on the special choice of defining sequence of $f(x)$.

Next, we set $\Phi_2$ the class of functions which can be expressed by difference of two functions of $\Phi_1$. Its integral is defined as follows:

If $f \in \Phi_2$ is expressed as $f(x) = f_1(x) - f_2(x)$, $f_1 \in \Phi_1$, $f_2 \in \Phi_1$ then:

$$\int f(x)\,dx = \int f_1(x)\,dx - \int f_2(x)\,dx.$$

$\Phi_2$ makes a vector space, and includes $\Phi_1$ and $\Phi$, and there their integrals coincide. If $f(x) \in \Phi_2$, $f(x) \geqq 0$ then $\int f(x)\,dx \geqq 0$. If $f \in \Phi_2$ then $f^+$, $f^-$ and $|f| \in \Phi_2$.

Further we can prove the Beppo Levi's theorem.

Every increasing sequence $\{h_n(x)\}$ ($h_n \in \Phi_2$, $n=1,2,\cdots$) whose integrals have common bound converges almost everywhere to a limit function $h \in \Phi_2$ and integration can be carried out term by term.

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4) Cf. F. Riesz and B. Sz.-Nagy: Lecons d'Analyse Fonctionnelle, Académie des Sciences de Hongrie (1952).

As a corollary of this theorem.

Every series $\sum_{n=1}^{\infty} k_n(x)$ ($k_n \in \Phi_2$, $n=1,2,\cdots$) for which $\sum_{n=1}^{\infty} \int |k_n(x)|\, dx$ converges, converges itself almost everywhere to a function of $\Phi_2$, and the series can be integrated term by term.

Of course the affirming theorems of integrability of a limit function (Lebesgue's theorem, Fatou's lemma) are true. As an application of them we get:

If $f \in \Phi$ and $B$ is a compact set then $f \cdot \chi_B \in \Phi_2$. In fact it is sufficient to show that if $O \in \mathcal{O}$ then $\chi_{\bar{O} \frown B} \in \Phi_2$; in this case we can construct a sequence of open sets $O^{(n)}$, each of which is a union of a finite number of sets $O_i \in \mathcal{O}$, and $O^{(1)} \supseteqq O^{(2)} \supseteqq \cdots \supseteqq O^{(n)} \supseteqq \cdots \supseteqq \bar{O} \frown B$ and $m\{O^{(n)} - (\bar{O} \frown B)\} \leqq 2^{-n}$. Denoting by $f_n(x)$ the characteristic function of $O^{(n)}$, $\{f_n(x)\}$ makes a decreasing sequence of functions of $\Phi$ which tends almost everywhere to $\chi_{\bar{O} \frown B}$. Further, if $f \in \Phi_2$ and $B$ is a compact set then $f \cdot \chi_B \in \Phi_2$. And finally, if $f \in \Phi_2$ and $B$ is a Borel set then $f \cdot \chi_B \in \Phi_2$. In fact, the family of set $B$, for which the proposition is true contains all compact sets and makes a $\sigma$-ring, therefore contains all Borel sets.

As a preparation we shall add the last one which concerns Haar measure.

Let $B$ be a Borel set and $m(B)=a$ then for any value $b$, $a \geqq b \geqq 0$ there exists a Borel subset $B' \subseteq B$ and $m(B')=b$. In fact we can assume $a > b > 0$, we select a natural number $n$ such that $a - 1/n > b > 1/n$, there exist an open Borel set $U: U \ni e$, $m(U) < 1/n$, and a compact set $C: C \subseteq B$, $m(B-C) < 1/n$. Since $C$ is covered by a finite number of $U \cdot x$, $C$ contains a Borel set $D$, $b \geqq m(D) \geqq b - 1/n$. By the same way, we can find a sequence of Borel sets $D_n$, $D_n \subseteq D_{n+1}$, $\lim_{n\to\infty} m(D_n) = b$. Therefore $\bigcup_{n=1}^{\infty} D_n$ is a desired set.

§ 2. After this preparation has been established, we shall proceed to a construction of integrals, which is quite parallel to the note of Prof. K. Kunugi.[1]

First, we introduce into $\Phi$ (recall $\Phi$ is a vector space generated by characteristic functions of sets $O \in \mathcal{O}$) topology and rank[5] so that they make $\Phi$ a uniform space and in the same time a ranked space. When positive integer or zero $\nu$ and a closed set $F \subseteq V$ are given we define a neighbourhood of the identically zero function 0, $v(F, \nu; 0)$ as the totality of functions $f(x)$ of $\Phi$ each of which has the following property: $f(x)$ is a sum of two functions of $\Phi$:

$$f(x) = p(x) + r(x)$$

and they satisfy the following conditions:

5) See 2).

[ 1 ] $r(x)$ vanishes for all $x \in F$.

[ 2 ] We have $\int |p(x)| dx < 2^{-\nu}$.

[ 3 ] We have $\left| \int r(x) dx \right| < 2^{-\nu}$.

The neighbourhood of a function $f \in \Phi$, $v(F, \nu; f)$ is defined as the totality of functions $g \in \Phi$ such that $g(x) - f(x) \in v(F, \nu; 0)$.

We can find without difficulty that the neighbourhoods just defined satisfy the following propositions.

(1*) All neighbourhoods $v(F, \nu; 0)$ contain the function 0.

(2*) If two arbitrary neighbourhoods of 0, $v(F_1, \nu_1; 0)$ and $v(F_2, \nu_2; 0)$ are given there exist neighbourhoods $v(F_3, \nu_3; 0)$ such that $v(F_3, \nu_3; 0) \subseteq v(F_1, \nu_1; 0) \frown v(F_2, \nu_2; 0)$.

(3*) For every neighbourhood of 0, $v = v(F, \nu; 0)$, we have $v = v^{-1}$.[6]

(4*) For any neighbourhood of 0, $v = v(F, \nu; 0)$, there exist neighbourhoods of 0, $w = v(F', \nu'; 0)$ such that $w^2 \subseteq v$.[6]

(5*) If $f \in \Phi$ is not identically zero, there exists a neighbourhood of 0, $v(F, \nu; 0)$, which does not contain the function $f$.

These propositions show that $\Phi$ is a uniform space.

To define the rank, we shall remark that the sequence of neighbourhoods $v(V, \nu; 0)$ $(\nu = 0, 1, 2, \cdots)$ is maximal monotone sequence.[7] Therefore the depth of the space $\Phi$ is $\omega_0$. The class $\mathfrak{B}_\nu$ of neighbourhoods of rank $\nu$ $(\nu = 0, 1, 2, \cdots)$ is defined as the totality of neighbourhoods $v(F, \nu; f)$, $f \in \Phi$ which satisfy the condition

$$m(V - F) < 2^{-\nu}.$$

Then, we can find that for any neighbourhood of $f$, $v = v(F, \nu; f)$ and for any rank $\mu$, there exists a neighbourhood $u$ of $f$ such that $u$ is contained in $v$ and the rank of $u$ is higher than $\mu$. Consequently, $\Phi$ is a ranked space.

We can introduce the notion of fundamental sequence and maximal collections quite similar to Prof. Kunugi.[1]

These notions are established, we can prove the following theorems:

THEOREM 1. *Let* $u \equiv \{u_n = v(F_n, \nu_n; f_n)\}$ *be a fundamental sequence. Then the functions* $f_n = f_n(x)$ *tend almost everywhere in* $V$ *to a function* $f(x)$.

THEOREM 2. *Let* $u \equiv \{u_n(f_n)\}$, $v \equiv \{v_n(g_n)\}$ *be fundamental sequences which belong to the same maximal collection. Set*

$$f(x) = \lim_{n \to \infty} f_n(x), \quad g(x) = \lim_{n \to \infty} g_n(x).$$

*Then, we have almost everywhere* $f(x) = g(x)$.

Therefore, if we identify two functions different only on a set of

6) $v^{-1}$ denotes the set of all functions $-f$ such that $f \in v$. $w^2 = w \cdot w$ denotes the set of all functions $f = g + h$ such that $g \in w$, $h \in w$.

7) Cf. 2).

measure 0, each maximal collection $f^*$ decides a function. We denote this function $J[f^*]$ and we shall call it a function associated to maximal collection $f^*$.

PROPOSITION 1. *Let* $u\equiv\{u_n(f_n)\}$ *be an arbitrary fundamental sequence. Then* $\int f_n(x)\,dx$ *forms a Cauchy sequence of real numbers.*

Consequently we can write: $I[u]=\lim_{n\to\infty}\int f_n(x)\,dx$.

PROPOSITION 2. *If two fundamental sequences* $u\equiv\{u_n(f_n)\}$, $v\equiv\{v_n(g_n)\}$ *belong to the same maximal collection* $f^*$ *then* $I[u]=I[v]$.

Therefore we can write this common value $I=I[f^*]$.

PROPOSITION 3. *Let* $u\equiv\{u_n=v(F_n,\nu_n;f_n)\}$ *be a arbitrary fundamental sequence, such that* $F_n\subseteq F_{n+1}$ $(n=0,1,2,\cdots)$, *and we set* $f(x)=\lim_{n\to\infty}f_n(x)$. *Then for every* $m$, $m=0,1,2,\cdots$ *the function* $f(x)\cdot\chi_{F_m}(x)$ *belongs to the class* $\Phi_2$.

In fact, the sequence of functions $(f_{n'}-f_n)^+\chi_{F_m}$ $(n>m, n'=n, n+1,\cdots)$, each of which belongs to $\Phi_2$, tends almost everywhere to $(f-f_n)^+\chi_{F_m}$, furthermore the sequence of values satisfies $\int(f_{n'}-f_n)^+\chi_{F_m}dx<2^{-\nu_n}$ $(n'=n,n+1,\cdots)$, and then by Fatou's lemma we have $(f-f_n)^+\chi_{F_m}\in\Phi_2$. Similarly we get $(f-f_n)^-\chi_{F_m}\in\Phi_2$. Therefore $f\cdot\chi_{F_m}\in\Phi_2$.

Let us consider a following property of the fundamental sequence $u\equiv\{v(F_n,\nu_n;f_n)\}$ — in the following we shall call it "property (P)".

(P) There exists a function of $n$ $(n=0,1,2,\cdots)$ $\phi(n)$ satisfying the following conditions:

(1) $\phi(n)>0$ for $n=0,1,2,\cdots$ .

(2) $\lim_{n\to\infty}\phi(n)=0$.

(3) For every Borel set $E$ contained in $V$ and whose measure does not exceed the measure of $V-F_n$, we have $\int|f_n(x)|\chi_E(x)dx\leqq\phi(n)$.

(4) $F_0\subseteq F_1\subseteq\cdots\subseteq F_n\subseteq\cdots$ .

(5) $\nu_0<\nu_1<\cdots<\nu_n<\cdots$ .

Then we can prove

THEOREM 3. *Each fundamental sequence* $u\equiv\{u_n=v(F_n,\nu_n;f_n)\}$ *which has the property (P) permit to define* $I[u]$ *as a limit of sequence of integrals*:

$$I[u]=\lim_{m\to\infty}\int f(x)\chi_{F_m}(x)\,dx,\quad f(x)=\lim_{n\to\infty}f_n(x).$$

Finally we say that a fundamental sequence $u\equiv\{u_n=v(F_n,\nu_n;f_n)\}$ has the property (P*) if it satisfies, in addition to the property (P), the following condition.

(6) There exists a positive integer $k$, $k\geqq2$ (independent of $n$)

which satisfies, for every $n$, $n=0, 1, 2, \cdots$, the inequality:

$$k \cdot m(V-F_{n+1}) \geqq m(V-F_n).$$

If we denote $\mathfrak{G}$ the set of all maximal collection $g^*$ each of which contains at least one fundamental sequence having the property (P*).

Then we can prove

THEOREM 4. *In the class* $\mathfrak{G}$, *the functions* $J[g^*]$ $(g^* \in \mathfrak{G})$ *form a vector space* (*over the field of real numbers*).

Theorem 3 and Theorem 4 together show that *in the class* $\mathfrak{G}$, *the number* $I[g^*]$ *is determined not only by* $g^*$ *but also by the function* $J[g^*]$.

Set $J[g^*]=f(x)$. We can write

$$I[g^*]=\int f(x)\,dx.$$

Thus we can construct a new integral for functions defined on a topological group.