58. *Division Problem of Some Species of Distributions*

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In this paper we shall show the following theorems. Theorem 1 was obtained by L. Ehrenpreis,*) but we give here a shorter proof.

Theorem 1. *Let Δ be any partial differential operator with constant coefficients. Then, for any distribution S there exists a distribution T such that $\Delta T=S$.*

Theorem 2. *Let Δ be as above. Then, for any distribution S of order k there exists a distribution T of order $k+2\left[\frac{n+2}{2}\right]+4$ such that $\Delta T=S$, where $[\ \ ]$ is the Gaussian symbol and n is the dimension of the underlying Euclidean space.*

Theorem 3. *Let Δ be as above. Then, for any locally square summable function f there exists a locally square summable function g such that $\Delta g=f$.*

1. **Preliminary notions.** By dx we denote the usual measure on R^n divided by $(2\pi)^{\frac{n}{2}}$. For any function $\varphi\in\mathcal{D}$, we define its Fourier transform $\Phi=\mathcal{F}(\varphi)$ by

$$\Phi(z)=\int\varphi(x)\,e^{-\sqrt{-1}x\cdot z}\,dx,$$

where $x\cdot z=x_1z_1+\cdots+x_nz_n$. We shall use lower case letters for functions of $\mathcal{D}$ and the corresponding upper case letters for their Fourier transforms.

Let us denote by D the set of all entire functions of exponential type which are rapidly decreasing on R^n. As is known, by the Paley-Wiener's theorem, D also can be characterized as the Fourier image of the set $\mathcal{D}$. We introduce a topology of D as follows. Let D_l be the set of all entire functions of exponential type $\leq l$ which are rapidly decreasing on R^n. Then $D=\bigcup_{l=1}^{\infty}D_l$. On D_l we give the topology defined by semi-norms

$$\nu_P(\Phi)=\sup_{z\in R^n}|P(z)\Phi(z)|,$$

where $P(z)$ denotes any polynomial on C^n. Then we can define the topology of inductive limit of the spaces D_l for $l=1,2,\cdots$. We give this topology on D. Then we can easily show that the Fourier transform is a topological isomorphism of $\mathcal{D}$ onto D.

*) L. Ehrenpreis: The division problem for distributions, Proc. Nat. Acad. Sci., **41**, 757–758 (1955); Solution of some problems of division, Amer. J. Math., **76**, 888–903 (1954).

2. Proof of Theorem 1 and Theorem 2. Let us denote by $Q(z)$ the Fourier image of Δ' which is the adjoint of the differential operator Δ. $Q(z)$ is a polynomial on C^n. By a linear coordinate transformation with real coefficients and positive Jacobian $z=\theta(w)$, $Q(z)$ can be written in the form

$$Q(z)=\alpha w_1^m+Q_0(w),$$

where $|\alpha|=1$ and m is the order of $Q(z)$ and $Q_0(w)$ is of order $\leq m-1$ with regard to w_1. So we assume, without loss of generality, that $Q(z)$ has this form.

Let F_j, $j=0, \pm 1, \cdots, \pm m$, be the set of all $z_1=u_1+\sqrt{-1}\,v_1$ where $|v_1-j|<1$. Then for any fixed $(z_2, \cdots, z_n)\in R^{n-1}$, $Q(z)$ is a polynomial of z_1 which has at most m distinct roots, and consequently there exists at least an integer j, $-m\leq j\leq m$, such that no roots of this polynomial belong to F_j. When j is given, we call here such a point $(z_2, \cdots, z_n)\in R^{n-1}$ a j-th point of R^{n-1}. Let E_m be the set of all m-th points of R^{n-1}, and E_{m-1} be the $(m-1)$-th points of R^{n-1} each of which does not belong to E_m, and so on. Thus we obtain the family of $2m+1$ sets $\{E_j \mid j=0, \pm 1, \cdots, \pm m\}$ which is, as is easily seen, a measurable disjoint covering of R^{n-1}, and for any j we have

$$\inf_{(z_1, z_2, \cdots, z_n)\in(R^1+\sqrt{-1}\,j)\times E_j} |Q(z)|\geq 1.$$

We give here the following lemma on one variable.

Lemma 1. *There exists a positive number $M=M_l$ such that, for any entire function $\Psi(z)$ of exponential type $\leq l$ which is rapidly decreasing on R^1, we have*

$$|\Psi(z)|\leq Me^{l|v|}\sup_{u\in R^1}|(1+u^2)\Psi(u)|,$$

where $z=u+\sqrt{-1}\,v$ as usual.

Proof. By the Paley-Wiener's theorem, the Fourier inverse image ψ of Ψ has its carrier in the interval $[-l, l]$. So we have

$$\begin{aligned}|\Psi(z)|&=\left|\int_{-l}^{l}\psi(x)\, e^{-\sqrt{-1}x(u+\sqrt{-1}v)}dx\right|\\ &\leq\frac{2l}{\sqrt{2\pi}}e^{l|v|}\sup_{x\in R^1}|\psi(x)|\\ &\leq Me^{l|v|}\sup_{u\in R^1}|(1+u^2)\psi(u)|.\end{aligned}$$

Though we deal with the case of many variables, we need only this lemma on one variable. Hereafter we go back to the case of many variables.

Let $Q(z)$ be as above, and QD be the set of all $Q\Phi$ for $\Phi\in D$ with the topology induced by D.

Lemma 2. *The map τ; $Q\Phi\to\Phi$ is a continuous linear map of QD onto D.*

Proof. The uniqueness and the linearity of this map are trivial, Moreover, owing to the topology of QD, it is sufficient to show the continuity of the map τ restricted on QD_l. Let $P(z)$ be any polynomial on C^n. Then for any $\Phi \in D_l$ we have

$$\begin{aligned}
&\sup_{z\in R^n} |P(z)\Phi(z)| = \max_{j=0,\cdots,\pm m} \{ \sup_{(z_1,z_2,\cdots,z_n)\in R^1\times E_j} |P(z)\Phi(z)|\} \\
&\leq \max_{j=0,\cdots,\pm m} \{Me^{l|j|} \sup_{(z_1,z_2,\cdots,z_n)\in(R^1+\sqrt{-1}j)\times E_j} |(1+z_1^2)P(z)\Phi(z)|\} \\
&\leq \max_{j=0,\cdots,\pm m} \{Me^{l|j|} \sup_{(z_1,z_2,\cdots,z_n)\in(R^1+\sqrt{-1}j)\times E_j} |(1+z_1^2)P(z)Q(z)\Phi(z)|\} \\
&\leq \max_{j=0,\cdots,\pm m} \{Me^{2l|j|} \sup_{(z_1,z_2,\cdots,z_n)\in R^1\times E_j} |(1+z_1^2)^2P(z)Q(z)\Phi(z)|\} \\
&\leq M^2e^{2lm} \sup_{z\in R^n} |(1+z_1^2)^2P(z)Q(z)\Phi(z)|.
\end{aligned}$$

This shows the continuity of the restricted map.

By the Fourier inverse transform, Lemma 2 can be interpreted as follows.

Lemma 3. *Let $\Delta'\mathcal{D}$ be the set of all $\Delta'\varphi$ for $\varphi \in \mathcal{D}$ with the topology induced by $\mathcal{D}$. Then the map $\Delta'\varphi \to \varphi$ is a continuous linear map of $\Delta'\mathcal{D}$ onto $\mathcal{D}$.*

Now we give the proof of Theorem 1.

By Lemma 3 and $S \in \mathcal{D}'$, the map $\tilde{T}$; $\Delta'\varphi \to S\cdot\varphi$ is a continuous linear functional on $\Delta'\mathcal{D}$. When we denote by T a continuous linear extension of $\tilde{T}$ to $\mathcal{D}$, T is certainly a distribution and for any $\varphi \in \mathcal{D}$ we have

$$\Delta T\cdot\varphi = T\cdot\Delta'\varphi = \tilde{T}\cdot\Delta'\varphi = S\cdot\varphi.$$

Theorem 2 follows easily from the topology of $\mathcal{D}^k$ and the above arguments.

3. **Proof of Theorem 3.** At the beginning, we introduce two spaces. Let us denote by lL_2 the set of all locally square summable functions, with the topology defined by the semi-norms

$$\mu_K(f) = \left\{ \int_K |f(x)|^2 dx \right\}^{\frac{1}{2}}$$

where K denotes any compact set in R^n. Next, we shall denote by cL^2 the set of all square summable functions with compact carriers, on which we define the topology as follows. Let K_l be a cube in R^n with center at the origin and side-length $2l$, and L_l^2 be the set of all functions in cL^2 whose carriers are contained in K_l. Then $cL^2 = \bigcup_{l=1}^{\infty} L_l^2$. On L_l^2 we give the topology defined by the usual L^2-*norm*, then on cL^2 we can define the topology of inductive limit of L_l^2 for $l=1, 2, \cdots$. We give this topology on cL^2.

Then, as is easily seen by using the Radon-Nikodym's theorem, we have

Lemma 4. *The dual of cL^2 is lL^2.*

$\mathscr{D}$ is dense in cL^2. By $\mathscr{D}_{cL^2}$ we denote the set $\mathscr{D}$ with the topology induced by the topology of cL^2. And by D_{cL^2} we denote the set D with the following topology. By $D_{L^2_l}$ we denote the set D_l with the topology defined by the usual L^2-*norm*, and on the set D we introduce the topology of inductive limit of $D_{L^2_l}$ for $l=1, 2, \cdots$.

Then it is easy to see that the Fourier transform is a topological isomorphism of $\mathscr{D}_{cL^2}$ onto D_{cL^2}.

Corresponding to Lemma 1, we have easily

Lemma 5. *In case of one variable, for any entire function Ψ of exponential type $\leq l$ which is rapidly decreasing on R^1, we have*

$$\left\{\int_{R^1} |\Psi(z)|^2 du\right\}^{\frac{1}{2}} \leq e^{l|v|} \left\{\int_{R^1} |\Psi(u)|^2 du\right\}^{\frac{1}{2}},$$

where $z=u+\sqrt{-1}\,v$.

Going back to the case of many variables again, we have

Lemma 6. *Let QD_{cL^2} be the set of all $Q\Phi$ for $\Phi \in D$ with the topology induced by D_{cL^2}. Then the map $Q\Phi \rightarrow \Phi$ is a continuous linear map of QD_{cL^2} onto D_{cL^2}.*

Thus by the similar arguments with the proof of Theorem 1, we can obtain Theorem 3.

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