82. On a Theorem of W. Sierpiński and S. Ruziewicz

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In my Note [1], we have generalized a theorem of W. Sierpiński [3]. In this Note we shall prove a theorem of S. Ruziewicz [2] and consider the relation of my result and his theorem. My result [1] is stated as follows: *Let M be an ordered set with power m. For a power n, $n \geq m$, if and only if the following proposition is true: for every element a of M, we can assign a family $\mathfrak{F}(a)$ of intervals such that each interval of it has a as end point and $\overline{\overline{\mathfrak{F}(a)}} < n$, and one of any distinct element of M is an end point of an interval of some $\mathfrak{F}(a)$.*

For an ordered set M with power m, let us consider the product space $M \times M$, then $A = \{(x, y) \mid y \in \mathfrak{F}(x)\} \cup \{(x, x) \mid x \in M\}$ and $B = \{(x, y) \mid x \in \mathfrak{F}(y)\}$ are disjoint. Further $A \smile B = M \times M$, therefore the set A, B gives a partition of $M \times M$. Hence the section $A(x_0)$ of A by a given x_0 has the power $< n$. On the other hand, the section $B(y_0)$ of B by any y has the power $< n$. Thus we have the following

Proposition. *Let M be an ordered set with power m. If $m \leq n$, then the product space $M \times M$ is decomposed into two sets A and B such that A meets with power $< n$ on every parallel line to the second coordinate axis and B meets with power $< n$ on every parallel line to the first coordinate axis.*

We shall prove the converse of the proposition. To prove that $m \leq n$, suppose that the set A, B is a partition of $M \times M$, and A, B satisfy the condition mentioned. Then we define $\Phi(a)$ as the set $\{y \mid (a, y) \in A, a \neq y\} \cup \{x \mid (x, a) \in B, x \neq a\}$. Therefore we have $\overline{\overline{\Phi(a)}} < n$, and for each a of M, we may define $\Phi(a)$. If x and y are distinct elements of M, then, by $(x, y) \in M$, $(x, y) \in A$ or $(x, y) \in B$. If $(x, y) \in A$, then $x \in \Phi(y)$, and if $(x,y) \in B$, then $y \in \Phi(x)$. Let us define $\mathfrak{F}(a)$ as all intervals (a, x) such that $x \in \Phi(a)$. It is obvious that $\overline{\overline{\mathfrak{F}(a)}} < n$, and one of distinct elements is an end point of an interval of type $\mathfrak{F}(a)$.

Therefore we have the following

Theorem. *Let M be an ordered set with power m. A power n is not less than m, if and only if the following statement: the product space $M \times M$ is decomposed into two disjoint sets such that one meets with the power $< n$ on each parallel line to the first coordinate axis and the other meets with power $< n$ on each parallel line to the second coordinate axis.*

Such a theorem was stated by S. Ruziewicz [2] and a special case related to the continuum hypothesis was stated by W. Sierpiński [3, 4] and [5, p. 376].

References

[1] K. Iséki: A generalisation of a theorem of W. Sierpiński, Proc. Japan Acad., **34**, 28 (1958).

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[3] W. Sierpiński: Hypothèse du Continu, Warszawa-Lwow (1934).

[4] W. Sierpiński: Sur une propriété de la droite équivalente à l'hypothèse du continu, Ganita, **5**, 113–116 (1954).

[5] W. Sierpiński: Cardinal and Ordinal Numbers, Warszawa (1958).