81. Relations between Solutions of Parabolic and Elliptic Differential Equations

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In this note we shall show that under some conditions the solution $u(x, t)$ of

$$\sum_{i=1}^{m} \frac{\partial^2 u}{\partial x_i^2} - \frac{\partial u}{\partial t} = f(x, t, u)$$

converges to a solution $v(x)$ of

$$\sum_{i=1}^{m} \frac{\partial^2 v}{\partial x_i^2} = \bar{f}(x, v)$$

as $t \to \infty$.

Let G be a domain which is regular for Laplace's equation[1] in the m-dimensional Euclidean space, and let Γ be the boundary of G. Set $D = G \times (0, \infty)$ and $B = \Gamma \times [0, \infty)$. We remark that D is regular for the heat equation[2] and therefore regular for the equation (E_1) below.[3]

Now, let $\mho$ and $\triangle$ be the generalized heat operator[4] and the generalized Laplacian operator respectively, i. e.

$$\mho u(x, t) = \lim_{r \downarrow 0} \frac{(n+2)^{\frac{m}{2}+1}}{m\pi^{\frac{m}{2}} r^2} \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cdots \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \{u(\xi, \tau) - u(x, t)\} \sin^{m-1}\theta$$
$$\times \cos\theta \, (\log \operatorname{cosec} \theta)^{\frac{m}{2}} \boldsymbol{J} d\varphi_1 \cdots d\varphi_{m-1} d\theta$$

and

$$\triangle u(x) = \lim_{r \downarrow 0} \frac{2 \cdot \Gamma\left(\frac{m}{2}+1\right)}{\pi^{\frac{m}{2}} r^2} \int_0^{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cdots \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \{u(\xi) - u(x)\} \boldsymbol{J} d\varphi_1 \cdots d\varphi_{m-1},$$

where in the first expression, $(\xi, \tau) = (\xi_1, \cdots, \xi_m, \tau)$ with

$$\xi_i = x_i + 2r\sqrt{m} \sin\theta \sqrt{\log \operatorname{cosec} \theta} \, \eta_i \qquad (i = 1, \cdots, m)$$

1) This means that the 1st boundary value problem of Laplace's equation for G is always solvable for any continuous data on Γ.

2) "Regular for the heat equation" means that the 1st boundary value problem of the heat equation for D is always solvable for any continuous data on $G \smile B$. D is regular for the heat equation if and only if G is regular for Laplace's equation. For the proof, see "On the regularity of domains for parabolic equations", Proc. Japan Acad., **34**, 347-348 (1958).

3) It was proved in [1, p. 626] that a p-domain is regular for (E_1) if and only if it is regular for the heat equation.

4) See [1, p. 627], in which we used the symbol □ instead of $\mho$.

$$\tau = t - r^2 \sin^2\theta \qquad \left(0 \le \theta \le \frac{\pi}{2}\right)$$

and in the second expression, $(\xi) = [\xi_1, \cdots, \xi_m)$ with

$$\xi_i = x_i + r\eta_i \qquad (i = 1, \cdots, m).$$

In both cases,

$$\eta_1 = \cos\varphi_1 \cos\varphi_2 \cdots\cdots\cdots \cos\varphi_{m-2} \cos\varphi_{m-1}$$
$$\eta_2 = \cos\varphi_1 \cos\varphi_2 \cdots\cdots\cdots \cos\varphi_{m-2} \sin\varphi_{m-1}$$
$$\eta_3 = \cos\varphi_1 \cos\varphi_2 \cdots\cdots\cdots \sin\varphi_{m-2}$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\eta_{m-1} = \cos\varphi_1 \sin\varphi_2$$
$$\eta_m = \sin\varphi_1 \qquad \left(-\frac{\pi}{2} \leqq \varphi_i \leqq \frac{\pi}{2},\ i = 1, \cdots, m-2;\ 0 \leqq \varphi_{m-1} \leqq 2\pi\right)$$

and

$$\boldsymbol{J} = \det \begin{vmatrix} \eta_1 & \eta_2 & \cdots\cdots & \eta_m \\ \dfrac{\partial\eta_1}{\partial\varphi_1} & \dfrac{\partial\eta_2}{\partial\varphi_1} & \cdots\cdots & \dfrac{\partial\eta_m}{\partial\varphi_1} \\ \cdots & \cdots & \cdots & \cdots \\ \dfrac{\partial\eta_1}{\partial\varphi_{m-1}} & \dfrac{\partial\eta_2}{\partial\varphi_{m-1}} & \cdots\cdots & \dfrac{\partial\eta_m}{\partial\varphi_{m-1}} \end{vmatrix}.$$

These operators have the following properties:

(i) If $u(x,t)$ and $u(x)$ are functions in the class C^2,

$$\mho\, u(x,t) = \sum_{i=1}^{m} \frac{\partial^2 u(x,t)}{\partial x_i^2} - \frac{\partial u(x,t)}{\partial t}$$

and

$$\triangle u(x) = \sum_{i=1}^{m} \frac{\partial^2 u(x)}{\partial x_i^2}.$$

(ii) If we operate $\mho$ to a function $u(x)$ which *does not* depend on t, we have

$$\mho\, u(x) = \triangle u(x).$$

Consider the following two equations:

$$(\mathrm{E}_1) \qquad \mho\, u = f(x,t,u) \qquad x \in G,\ t \ge 0,$$

$$(\mathrm{E}_2) \qquad \triangle v = \bar{f}(x,v) \qquad x \in G,$$

where $f(x,t,u)$ and $\bar{f}(x,v)$ are continuous functions on $D \times (-\infty, \infty)$ and $G \times (-\infty, \infty)$ respectively, quasi-bounded[5] with respect to u and v and non-decreasing with respect to u and v.

Let $g(x)$ be a continuous function on $G \smile \Gamma$ and $\varphi(\bar{x}, t)$ be a continuous function on B and moreover $\varphi(\bar{x}, 0) = g(\bar{x})$ for $\bar{x} \in \Gamma$. Let $u(x,t)$ be a solution[6] of (E_1) which is continuous on $D \smile G \smile B$ and which satisfies the boundary condition $u(x,0) = g(x)$ $(x \in G)$ and $u(\bar{x},t) = \varphi(\bar{x},t)$ $(x \in \Gamma,\ t \ge 0)$. Assume that $\varphi(\bar{x},t)$ converges uniformly on Γ to a

5) We say that a function $f(p,q)$ defined on $E \times F$ is quasi-bounded with respect to q if $f(p,q)$ is bounded on $E \times K$, where K is any compact set in F.

6) 7) These solutions $u(x,t)$ and $v(x)$ *do exist*. See [1] and [2].

function $\varphi(\bar{x})$ as $t\to\infty$. (Then $\varphi(\bar{x})$ is again a continuous function on Γ.) Let $v(x)$ be a solution[7] of (E_2) which is continuous on $G\smile\Gamma$ and satisfies $v(\bar{x})=\varphi(\bar{x})$ on Γ.

Finally assume that, for any $U>0$, $f(x,t,u)$ converges uniformly to $\bar{f}(x,u)$ on the set $\{(x,u);\ x\in G,\ |u|\leq U\}$ as $t\to\infty$.[8]

Under these assumptions, $u(x,t)$ converges uniformly to $v(x)$ on $G\smile\Gamma$ as $t\to\infty$.

Proof. For any $\varepsilon>0$, there exists $T_1>0$ such that $|\varphi(\bar{x},t)-\varphi(\bar{x})|<\varepsilon$ for $t\geq T_1$. Set $M_0=\max\{|g(x)|;\ x\in G\smile\Gamma\}$, $M_1=\max\{|\varphi(\bar{x},t)|;\ x\in\Gamma,\ 0\leq t\leq T_1\}$ and $M_2=\max\{|v(x)|;\ x\in G\smile\Gamma\}$. By the assumption above we can find a constant $T_2>0$ such that

$$|f(x,t,v(x))-\bar{f}(x,v(x))|<\varepsilon$$

for $x\in G$, $t\geq T_2$. Set $M_3=\sup\{|f(x,t,v(x))-\bar{f}(x,v(x))|;\ x\in G,\ 0\leq t\leq T_2\}$. Let $\psi(x)$ be a solution of $\triangle\psi=-1$ such that $\psi(x)$ is continuous on $G\smile\Gamma$ and vanishes on Γ. Then there exists a constant Ψ such that $0\leq\psi(x)\leq\Psi$, hence we can take a constant $\alpha>0$ such that $-1+\alpha(1+\Psi)<-\frac{1}{2}$. Finally, let $M>0$ be a constant such that (i) $\frac{1}{2}Me^{-\alpha T_2}>M_3$, (ii) $Me^{-\alpha T_1}>M_1+M_2$ and (iii) $M>M_0+M_2$.

Consider the function $Me^{-\alpha t}+\varepsilon$. Then, we have

$$|\varphi(\bar{x},t)-\varphi(\bar{x})|<Me^{-\alpha t}+\varepsilon \qquad x\in\Gamma,\ t\geq 0.$$

Now, let $v_1(x,t)$ be a solution of the equation:

$$\sum_{i=1}^{m}\frac{\partial^2 v}{\partial x_i^2}=-Me^{-\alpha t}-\varepsilon,$$

and suppose that $v_1(x,t)$ is continuous on $G\smile\Gamma$ and admits the boundary value $Me^{-\alpha t}+\varepsilon$ on Γ. Then we have

$$\begin{aligned}v_1(x,t)&=Me^{-\alpha t}+\varepsilon+\psi(x)(Me^{-\alpha t}+\varepsilon)\\&=Me^{-\alpha t}(1+\psi(x))+\varepsilon(1+\psi(x))\\&=(Me^{-\alpha t}+\varepsilon)(1+\psi(x)).\end{aligned}$$

Set $V(x,t)=v(x)+v_1(x,t)$, then

$$\begin{aligned}\mho V(x,t)&=\triangle v(x)+\sum_{i=1}^{m}\frac{\partial^2 v_1(x,t)}{\partial x_i^2}-\frac{\partial v_1(x,t)}{\partial t}\\&=\bar{f}(x,v(x))+(-Me^{-\alpha t}-\varepsilon)+\alpha Me^{-\alpha t}(1+\psi(x))\\&=\bar{f}(x,v(x))+\varepsilon+Me^{-\alpha t}(-1+\alpha(1+\psi(x))).\end{aligned}$$

Now, for $u>V(x,t)$ we have

$$f(x,t,u)-\mho V(x,t)\geq f(x,t,v(x))-\bar{f}(x,v(x))+\varepsilon-Me^{-\alpha t}(-1+\alpha(1+\psi(x))).$$

Since $f(x,t,v(x))-\bar{f}(x,v(x))>-M_3$ and $-Me^{-\alpha t}(-1+\alpha(1+\psi(x)))>M_3$ for $0\leq t\leq T_2$, we have

$$f(x,t,u)-\mho V(x,t)>0$$

for $0\leq t\leq T_2$. For $t\geq T_2$, since $f(x,t,v(x))-\bar{f}(x,v(x))>-\varepsilon$, we have

8) It is sufficient for our proof to assume that $f(x,t,v(x))$ converges uniformly to $\bar{f}(x,v(x))$ on G as $t\to\infty$.

$$f(x, t, u) - \triangledown V(x, t) > 0.$$

Consequently, if $u > V(x, t)$, $x \in G$ and $t > 0$, then we obtain

$$f(x, t, u) - \triangledown V(x, t) > 0.$$

Next, on the boundary B, since $\varphi(\bar{x}, t) \leq \varphi(\bar{x}) + Me^{-\alpha t} + \varepsilon$, we have

$$u(\bar{x}, t) = \varphi(\bar{x}, t) \leq \varphi(\bar{x}) + (Me^{-\alpha t} + \varepsilon)(1 + \psi(\bar{x})) = V(\bar{x}, t).$$

On G, the rest part of the boundary of D,

$$u(x, 0) = g(x) \leq M_0 < M - M_2 \leq v(x) + (1 + \psi(x))(M + \varepsilon)$$

implies $V(x, 0) \geq u(x, 0)$. Hence, on the whole boundary of D, we have

$$V(x, t) \geq u(x, t).$$

Therefore by the comparison theorem,[9] we have

$$u(x, t) \leq V(x, t) = v(x) + v_1(x, t)$$

on $D \cup G \cup B$. Similarly we have $v(x) - v_1(x, t) \leq u(x, t)$, and consequently

$$| u(x, t) - v(x) | \leq v_1(x, t)$$

on $D \cup G \cup B$.

Since $v_1(x, t) = (Me^{-\alpha t} + \varepsilon)(1 + \psi(x))$, there exists a constant $T_3 > 0$ such that $| v_1(x, t) | \leq 2(1 + \Psi)\varepsilon$ for $x \in G \cup \Gamma$ and $t \geq T_3$. Thus $u(x, t)$ converges uniformly to $v(x)$ on $G \cup \Gamma$. This completes the proof.

Corollary 1. Assume that moreover $f(x, t, 0) \equiv 0$. Then, the solution of (E_1) which admits $g(x)$ on G (where $g(\bar{x}) = 0$ for $\bar{x} \in \Gamma$) and which vanishes on B converges uniformly to zero on $G \cup \Gamma$.

This shows that the solution is asymptotically stable.

Corollary 2. If $\varphi(\bar{x}, t)$ converges uniformly to $\varphi(\bar{x})$ on Γ, the solution of the heat equation which admits $\varphi(\bar{x}, t)$ on B and which admits $g(x)$ on G converges uniformly to the solution of Laplace's equation which admits $\varphi(\bar{x})$ on Γ.

This means that the solution of the heat equation converges to the steady state solution.[10]

References

[1] H. Murakami: On non-linear partial differential equations of parabolic types. I-III, Proc. Japan Acad., **33**, 530-535, 616-627 (1957).

[2] T. Satō: Sur l'équation aux dérivées partielles $\Delta z = f(x, y, z, p, q)$, Comp. Math., **12** (1954) and Sur l'équation aux dérivées partielles $\Delta z = f(x, y, z, p, q)$ II (to appear). Also M. Hukuhara and T. Satō: Theory of Differential Equations (in Japanese), Kyōritsu Publ. Co., Ltd., Tokyo (1957).

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9) Theorem 2.1 [1, p. 533].

10) See also W. Fulks: A note on the steady state solution of the heat equation, Proc. Amer. Math. Soc., **7** (1956). He assumed that $\varphi(\bar{x}, t)$ is monotone increasing with t. This assumption plays essential role in his proof but our proof does not need it.