80. On the Regularity of Domains for Parabolic Equations

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Let G be a domain with the boundary Γ in the m-dimensional Euclidean space R^m. Let $D=G\times(0,\infty)$ and $B=\Gamma\times[0,\infty)$. W. Fulks pointed out in [1] by constructing barriers at every boundary point of G that *if D is regular for the heat equation then G is regular for Laplace's equation. In this note we shall show also by constructing barriers of the parabolic equation at every point of $G\smile B$ that the converse of the result above by W. Fulks is true.*

Consider the equation

(E) $$\triangledown u = f(x,t,u)$$ [1)]

where $f(x,t,u)$ is continuous on $D\times(0,\infty)$ and quasi-bounded with respect to u.

As in [2, p. 623], we say that $w(x,t)$ is *a barrier of* (E) *at a boundary point $(x^0,t^0)\in G\smile B$ with respect to a bounded function $\beta(x,t)$ defined on $G\smile B$* if $w(x,t)$ satisfies:

(i) $w(x,t)$ is continuous on $\overline{D}$,

(ii) $w(x,t)>0 \qquad (x,t)\in\overline{D},\quad (x,t)\neq(x^0,t^0)$,

(iii) $w(x,t)\to 0 \qquad (x,t)\to(x^0,t^0),\quad (x,t)\in\overline{D}$,

(iv) $\triangledown w(x,t)\leqq -M$, where

$$M=\sup\{|f(x,t,\bar{\beta}(x^0,t^0))|,\ |f(x,t,\underline{\beta}(x^0,t^0))|;\ (x,t)\in\overline{D}\}.$$

It is known[2)] that if every point of $G\smile B$ has barriers then D is regular for (E), i.e. the first boundary value problem of (E) is always solvable for any continuous data on $G\smile B$.

Now we shall construct the barrier $w(x,t)$ satisfying the conditions (i), (ii), (iii) and (iv) under the assumption that G is regular for Laplace's equation.

In case that $(x^0,t^0)\in G$, it is easy to see that the function $w(x,t)=\sum_{i=1}^{m}(x_i-x_i^0)^2+(2m+M)(t-t^0)$ is a barrier at (x^0,t^0). In case that $(x^0,t^0)\in B$ with $t^0>0$, let $\varphi(x,t)=\sum_{i=1}^{m}(x_i-x_i^0)^2+(t-t^0)^2$. Then we have $\varphi(x,t)\geqq 0$ and

$$\triangledown\varphi(x,t)=\sum_{i=1}^{m}\frac{\partial^2\varphi(x,t)}{\partial x_i^2}-\frac{\partial\varphi(x,t)}{\partial t}$$
$$=2(m-(t-t^0)).$$

1) $\triangledown$ and $\triangle$ below are respectively the generalized heat and Laplacian operators. For the definitions, see [2, p. 627], where $\triangledown$ is denoted by $\square$. See also [3, p. 349].

2) [2, pp. 624-626].

For a boundary vanishing solution $\psi(x)$ of $\triangle\psi=-2(m+t^0)-M$ on $G\smile\Gamma$, the function $w(x,t)=\psi(x)+\varphi(x,t)$ has the following properties:

(i) $w(x,t)$ is continuous on $\overline{D}$,

(ii) $w(x,t)>0 \qquad (x,t)\in\overline{D}, \quad (x,t)\neq(x^0,t^0)$,

(iii) $w(x,t)\to 0 \qquad (x,t)\to(x^0,t^0), \quad (x,t)\in\overline{D}$,

(iv)
$$\begin{aligned}\sqcup w(x,t)&=\sqcup\{\psi(x)+\varphi(x,t)\}\\&=\triangle\psi(x)+\sqcup\varphi(x,t)\\&=-2(m+t^0)-M+2\{m-(t-t^0)\}\\&=-M-2t\\&\leqq -M.\end{aligned}$$

Thus, $w(x,t)$ is a barrier of (E) at (x^0,t^0). This completes the proof.

References

[1] W. Fulks: A note on the steady state solution of the heat equation, Proc. Amer. Math. Soc., **7** (1956).

[2] H. Murakami: On non-linear partial differential equations of parabolic types. I-III, Proc. Japan Acad., **33**, 530-535, 616-627 (1957).

[3] H. Murakami: Relations between solutions of parabolic and elliptic differential equations, Proc. Japan Acad., **34**, 349-352 (1958).