75. *Hukuhara's Problem for Hyperbolic Equations with Two Independent Variables. I. Semi-linear Case*

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Introduction. In the Cauchy problem for partial differential equations involving N unknown functions u_i $(i=1,\cdots,N)$ of two independent variables, the initial values of the functions u_i are prescribed on *a curve*. Generalizing the way of giving the Cauchy conditions, we can set up another problem where *N curves* C_i are given and on each C_i the value of u_i is prescribed. This will be called *Hukuhara's problem*, or shortly, Problem H, since M. Hukuhara has originally studied an analogous problem for the system of ordinary differential equations.

Generally speaking, the solutions of the Cauchy problems for hyperbolic systems are *stable*, that is to say, the *slight* change of the initial values entails but a *slight* change of the solutions. In physical problems, the initial values of unknown quantities are found experimentally and can not be determined with absolute precision, so that the stability property of the problem has a special importance in this case. However, the errors of the measurement may be committed also *concerning the situation of the initial curves*, i.e. the values of the different unknown quantities can not be measured at the exactly same time and place. This consideration leads us to the concept of the Hukuhara's problem, and the influence of the change of the situation of the curves C_i upon the solutions should be investigated to secure the stability property of the problem.

We shall show in this paper, the *correctly posedness* of the Hukuhara's problem for semi-linear hyperbolic systems with two independent variables, which are assumed to be of diagonal form, since every semi-linear system can be transformed to this normal form by the well-known techniques. Same results for the quasi-linear systems will be stated in the following report entitled Part II. Our proof is given by means of the elementary method of successive approximation, since the stability of the solutions concerning the situation of the curves C_i, one of the characteristic properties of our problem, can be obtained by it very easily.

Recently, analogous problems have been treated by two Polish mathematicians, Z. Szmydt [2-5] and A. Pliś [1]. The former proved the existence and the uniqueness of solutions of a problem intimately connected with ours for the higher order hyperbolic systems with two independent variables, of which the characteristics were all straight lines parallel to one of the two axes. A. Pliś proved analogous results for a first order system which involves an arbitrary number of independent variables $(x, y_1, \cdots, y_m)$. In his paper, the values of unknowns were prescribed on the parallel hyperplanes: $x=a_i$ $(i=1, \cdots, N)$. Both authors have not discussed the stability of the solutions.

1. Problem H. The norm $\|g\|_D$ of a vector $g=(g_1, \cdots, g_N)$, whose components are functions defined on some domain D, will be defined as $\|g\|_D=\sup_i (\sup_D |g_i|)$.

Consider a real semi-linear hyperbolic system of the form

$$(1) \qquad \partial u_i/\partial t-\lambda_i(t, x)\partial u_i/\partial x=f_i(t, x, u), \quad i=1, 2, \cdots, N.$$

We make the following assumptions: $1°$ *λ_i are defined on a strip $G=\{0\leq t\leq B, B>0\}$ in (t, x)-space and belong to class C^1, $\|\lambda\|_G$, $\|\partial\lambda/\partial x\|_G$ being finite,* $2°$ *f_i are defined on a strip $\overline{G}=\{(t, x)\in G, \|u\|\leq\rho\}$ in (t, x, u)-space and belong to class C^1, where $u=(u_1, \cdots, u_N)$ and $\|u\|=\sup_i |u_i|$; $\|f\|_{\overline{G}}$, $\|\partial f/\partial x\|_{\overline{G}}$, $\|\partial f/\partial u_i\|_{\overline{G}}$ being finite.* Consequently f is Lipschitzian with respect to u, i.e. $|f_i(u)-f_i(v)|\leq L\cdot\|u-v\|$ with a constant L. Let θ, M_1, M_2 be constants such as $\pi/2\geq\theta>0$, $\rho>M_1>0$, $M_2>0$. We call the set of the constants $\{B, \rho, \theta, M_1, M_2, \|\lambda\|_G, \|\partial\lambda/\partial x\|_G, \|f\|_{\overline{G}}, \|\partial f/\partial x\|_{\overline{G}}, \|\partial f/\partial u_i\|_{\overline{G}}, i=1, 2, \cdots, N\}$ the *size* of Problem H.

Under these assumptions, there exists a positive constant B_0 determined by several inequalities which consist only of the quantities derived from the size of Problem H, and we can solve the problem on the strip $G_0=\{0\leq t\leq B_0\}$. More precisely; let *N smooth curves C_i* be situated on G_0 in such a way that the i-th curve C_i intersects every one of the whole family of the i-th characteristics of (1) at *one and only one point* and at *an angle not less than θ*. For every i let ϕ_i be a *smooth function* defined on the i-th curve C_i which satisfies the *inequalities* $|\phi_i|\leq M_1$, $|\partial\phi_i/\partial s_i|\leq M_2$, where $\partial/\partial s_i$ means the derivation along the curve C_i. Then we can find a solution of (1) which is defined on G_0, equal to ϕ_i on C_i, unique and depends continuously on curves C_i and functions ϕ_i. In other words, *the problem is correctly posed in the sense of Hadamard.*

2. Integration along the characteristics. The i-th characteristic l_i of (1), passing through a point (t_0, x_0) of G is expressed as $x=\psi_i(t)$ by the solution ψ_i of

$$(2) \qquad d\psi_i(t)/dt=-\lambda_i(t, \psi_i(t)), \quad \psi_i(t_0)=x_0.$$

By the theory of ordinary differential equations, $\{\|\partial\psi/\partial x_0\|_{0\leqq t\leqq B}\}_{(t_0, x_0)\in G}$ has a l.u.b., which can be determined by the size of Problem H. For the moment we impose on B_0 merely the condition such as $B_0\leq B$

and proceed by formal calculations. Further conditions for B_0, by which we can verify the results obtained, will be stated later. Let (t_0, x_0) be a point of G_0 and l_i be the i-th characteristic passing through (t_0, x_0). Take N curves C_i as described in § 1 and let (t_i, x_i) be the intersecting points of l_i and C_i for $i=1, 2, \cdots, N$. t_i and x_i become uniquely determined functions of (t_0, x_0) and belong to class C^1. As C_i and l_i intersect at an angle not less than θ, the boundedness of $\|\partial\psi_i/\partial x_0\|_{0\leq t\leq B}$ assures that $\|(\partial t_i/\partial x_0)\ i=1, 2, \cdots, N\|_{G_0}$ and $\|(\partial x_i/\partial x_0)\ i=1, 2, \cdots, N\|_{G_0}$ have l.u.b.'s determined by the size.

Integrating the i-th equation of (1) along the i-th characteristic l_i under the condition of $u_i(t_i, x_i)=\phi_i(t_i, x_i)$, we obtain

$$(3)\qquad u_i(t_0, x_0)=\phi_i(t_i, x_i)+\int_{t_i}^{t_0} f_i(t, \psi_i(t), u(t, \psi_i(t)))dt.$$

Conversely, if a solution of (3) belongs to class C^1, it must be the solution of Problem H. We shall find the solution of (3) by the successive approximation and prove its differentiability later.

We put $u_i^{(0)}(t_0, x_0)=\phi_i(t_i, x_i)$, $i=1, 2, \cdots, N$ and define $\{u^{(n)}\}$ successively by the formulas

$$(4)\qquad u_i^{(n+1)}(t_0, x_0)=\phi_i(t_i, x_i)+\int_{t_i}^{t_0} f_i(t, \psi_i(t), u^{(n)}(t, \psi_i(t)))dt.$$

The inequalities $\|u^{(n)}\|_{G_0}\leq M_1+M\cdot B_0$ and $\|u^{(n+1)}-u^{(n)}\|_{G_0}\leq M\cdot L^{n-1}\cdot B_0^n$ for every n can be proved easily, so that, if we impose on B_0 the conditions

$$(5)\qquad L\cdot B_0<1 \quad\text{and}\quad M_1+M\cdot B_0\leq\rho,$$

all formal calculations are justified. Namely, we have $\|u^{(n)}\|_{G_0}\leq\rho$ for every n and $u^{(n)}$ converges as $n\to\infty$ uniformly on G_0 to a limit function u, which is continuous and solves the integral equations (3).

3. **Uniqueness and stability of the solution.** The solution of (3), which is continuous and $\|u\|_{G_0}\leq\rho$, is unique, hence, the solution of Problem H enjoying the same properties is unique.

Indeed, if two such solutions u, v exist, we should have

$$u_i(t_0, x_0)-v_i(t_0, x_0)=\int_{t_i}^{t_0}[f_i(u)-f_i(v)]dt.$$

Hence $\|u-v\|_{G_0}\leq L\cdot B_0\cdot\|u-v\|_{G_0}$, so that $\|u-v\|_{G_0}$ must be zero, since we have imposed on B_0 the conditions (5).

We divide the proof of the stability of the solution into two parts.

1° Continuous dependence on ϕ_i. Take the solutions $u, \overline{u}$ of (3) corresponding to ϕ_i and $\overline{\phi}_i$ respectively, where ϕ_i and $\overline{\phi}_i$ are assumed to satisfy the condition such as $|\phi_i-\overline{\phi}_i|\leq\varepsilon$ on C_i for a constant $\varepsilon>0$, $i=1, 2, \cdots, N$. Then the 0-th approximations of $u, \overline{u}$ satisfy the inequality $\|u^{(0)}-\overline{u}^{(0)}\|_{G_0}\leq\varepsilon$, and we can prove by induction the formula

$\|u^{(n)}-\bar{u}^{(n)}\|_{G_0}\leq\varepsilon\cdot\sum_{k=0}^{n}(L\cdot B_0)^k$, so that the uniform convergence of the approximating sequences and the inequalities (5) yield $\|u-\bar{u}\|_{G_0}\leq\varepsilon/(1-L\cdot B_0)$. Hence $\bar{u}$ converges uniformly to u on G_0 as $\varepsilon\to 0$.

2° Continuous dependence on C_i. Let two sets of curves C_i and $\bar{C}_i$ be situated on G_0 as indicated in § 1. We define the one-to-one correspondence between the points of C_i and $\bar{C}_i$ such that the points of C_i and $\bar{C}_i$, situated on one and the same i-th characteristic, should be put into correspondence. We assume that for every i the curves C_i and $\bar{C}_i$ should be placed near to each other, i.e. the inequalities $|t_i-\bar{t}_i|\leq\varepsilon$, $i=1,2,\cdots,N$ should hold with some constant $\varepsilon>0$, where t_i and $\bar{t}_i$ mean the t-coordinates of the points (t_i, x_i) and $(\bar{t}_i, \bar{x}_i)$ which are put into correspondence as defined above. Let ϕ_i be given on C_i, then we define $\bar{\phi}_i$ on $\bar{C}_i$ by the relation $\bar{\phi}_i(\bar{t}_i, \bar{x}_i)=\phi_i(t_i, x_i)$. For the data C_i, ϕ_i and $\bar{C}_i$, $\bar{\phi}_i$ respectively, we have two solutions $u, \bar{u}$. As for their 1st approximations the inequality $\|u^{(1)}-\bar{u}^{(1)}\|_{G_0}\leq\max_i\left|\int_{t_i}^{\bar{t}_i}f_i\,dt\right|\leq\varepsilon\cdot M$ holds, and we can verify by induction, $\|u^{(n)}-\bar{u}^{(n)}\|_{G_0}\leq\varepsilon\cdot M\cdot\sum_{k=0}^{n-1}(B_0\cdot L)^k$ for every n, hence as the limit of $n\to\infty$ we have $\|u-\bar{u}\|_{G_0}\leq\varepsilon\cdot M/(1-B_0\cdot L)$, which asserts us the uniform convergence of $\bar{u}$ to u on G_0 as $\varepsilon\to 0$.

From 1° and 2° we know that the solution of (3), hence that of Problem H, depends on ϕ_i and C_i, $i=1,2,\cdots,N$ continuously, if the condition $\|u\|_{G_0}\leq\rho$ is imposed on it.

4. Differentiability of the solution. As $u=(u_1,\cdots,u_N)$ is the solution of (3), u_i is continuously differentiable along the i-th characteristic, hence, to prove the smoothness of u, it is sufficient to show that $\partial u_i/\partial x$ exists and is continuous on G_0 for every i.

1° $\|\partial u^{(n)}/\partial x_0\|_{G_0}$ is uniformly bounded for every n.

Differentiation of (4) with respect to x_0 yields

$$\frac{\partial u_i^{(n+1)}}{\partial x_0}=\frac{\partial\phi_i(t_i, x_i)}{\partial x_0}+f_i\cdot\frac{\partial t_i}{\partial x_0}+\int_{t_i}^{t_0}\left[\frac{\partial f_i}{\partial x}+\sum_{j=1}^{N}\frac{\partial f_i}{\partial u_j}\,\frac{\partial u_j^{(n)}}{\partial x}\right]\frac{\partial\psi_i}{\partial x_0}dt. \tag{6}$$

For a fixed constant K such as $K>\|(\partial\phi_i(t_i, x_i)/\partial x_0)_{i=1,2,\cdots,N}\|_{G_0}+\|f\|_{G_0}\cdot\|(\partial t_i/\partial x_0)_{i=1,2,\cdots,N}\|_{G_0}$ we have evidently $\|\partial u^{(0)}/\partial x_0\|_{G_0}\leq K$, and by induction the inequality $\|\partial u^{(n)}/\partial x_0\|_{G_0}\leq K$ can be proved for every n, provided that the condition such as

$$\begin{aligned}&\|(\partial\phi_i/\partial x_0)_{i=1,2,\cdots,N}\|_{G_0}+\|f\|_{G_0}\cdot\|(\partial t_i/\partial x_0)_{i=1,2,\cdots,N}\|_{G_0}\\&+B_0\|\partial\psi/\partial x_0\|_{G_0}\cdot\Big[\|\partial f/\partial x_0\|_{G_0}+K\cdot\sum_{j=1}^{N}\|\partial f/\partial u_j\|_{G_0}\Big]\leq K\end{aligned} \tag{7}$$

is satisfied by B_0.

2° $\partial u^{(n)}/\partial x_0$ converges uniformly on G_0.

From (6) we have

$$
\begin{aligned}
(8)\qquad \frac{\partial u_i^{(m+1)}}{\partial x_0}-\frac{\partial u_i^{(n+1)}}{\partial x_0} &= \frac{\partial t_i}{\partial x_0}[f_i(u^{(m)})-f_i(u^{(n)})]+\int_{t_i}^{t_0}\Big[\frac{\partial f_i(u^{(m)})}{\partial x} \\
&\quad -\frac{\partial f_i(u^{(n)})}{\partial x}\Big]\cdot\frac{\partial\psi_i}{\partial x_0}dt+\int_{t_i}^{t_0}\sum_{j=1}^{N}\Big[\frac{\partial f_i(u^{(m)})}{\partial u_j}-\frac{\partial f_i(u^{(n)})}{\partial u_j}\Big]\cdot\frac{\partial u_j^{(m)}}{\partial x}\cdot\frac{\partial\psi_i}{\partial x_0}dt \\
&\quad +\int_{t_i}^{t_0}\sum_{j=1}^{N}\frac{\partial f_i(u^{(n)})}{\partial u_j}\Big[\frac{\partial u_j^{(m)}}{\partial x}-\frac{\partial u_j^{(n)}}{\partial x}\Big]\cdot\frac{\partial\psi_i}{\partial x_0}dt.
\end{aligned}
$$

The 1st, 2nd and 3rd terms of R. S. converge uniformly to 0 on G_0 as $m, n\to\infty$, so that if we define d such as $d=\limsup_{m,n\to\infty}\{\|\partial u^{(m)}/\partial x_0-\partial u^{(n)}/\partial x_0\|_{G_0}\}$, we obtain the inequality

$$(9)\qquad d\leq B_0\cdot\|\partial\psi/\partial x_0\|_{G_0}\cdot\sum_{j=1}^{N}\|\partial f/\partial u_j\|_{G_0}\cdot d.$$

If we impose on B_0 the condition such that the coefficient of d in the R. S. of (9) should be less than one, then d must be equal to zero, so that $\partial u^{(n)}/\partial x_0$ converges uniformly on G_0. Therefore the limit function u of $u^{(n)}$ is continuously differentiable in x-direction, hence it belongs to class C^1 and gives the solution of Problem H.

5. Conditions for the finiteness of domain G. Hitherto we have discussed the problem on a strip $G=\{0\leq t\leq B\}$ which was assumed to be infinite in x-direction. In this section we shall give the several cases in which Problem H can be settled on a finite domain.

Divide G into two parts by a curve l_0 which is defined such as $x=\psi_0(t)$, $0\leq t\leq B$, and let $G'=\{x\geq\psi_0(t),\ 0\leq t\leq B\}$ be its right half part. If every point of G' can be connected to C_i by the i-th characteristic l_i without leaving G' for every i, we say that Problem H satisfies the *left finiteness condition along* l_0. Then §§ 1–4 are still valid if G is replaced with G'. The right finiteness condition is defined similarly.

1° If N characteristics unite and make up a curve l_0 which divides G into two parts, then the left and right finiteness conditions are satisfied along this l_0.

2° If all N curves C_i pass through a fixed point (t_0, x_0) of G and if there exists a piecewise smooth curve l_0 such as $x=\psi_0(t)$, $0\leq t\leq B$, which satisfies the following conditions: $x_0=\psi_0(t_0)$, $d\psi_0(t)/dt\leq -\lambda_i(t,\psi_0(t))$, for $t>t_0$, and $d\psi_0(t)/dt\geq-\lambda_i(t,\psi_0(t))$, for $t<t_0$, $i=1,2,\cdots,N$, then the right finiteness condition is satisfied along l_0. Similar argument is valid for the left finiteness condition.

Cauchy problems, regarded as the special case of our Problem H, satisfy the above condition 2° in a trivial way, and any straight line whose slope is $\pm M$, M is a constant such as $M\geq\|\lambda\|_G$, can play the part of the curve l_0.

References

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