73. On Some Existence Theorems on Multiplicative Systems. II. Maximal Subsystems*)

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§ 1. Introduction. We have given several existence theorems on multiplicative systems in the previous paper [1]. Here we shall be concerned in this short note with a part which shares a portion on one side of duality whose other side has been the main part of [1]. We shall use terminologies defined in [1] without mentioning.

In the second section we shall state existence theorems of the greatest subsystem, while in the fourth section we shall give those of maximal subsystems. Before presenting the latter we need to introduce the notion of amalgamated product, which appears new in general case and will be defined in the third section.

The fifth section is devoted to make a comparison between the notions defined and used in this note and those in [1]. The dual part of § 6 in the previous paper [1] is omitted here which will be given in the subsequent paper.

In this paper the empty set is considered as a system, unless otherwise specified. Thus for any system, the empty system is a subsystem of it. Also systems discussed in this paper are assumed to be multiplicative systems with the same set M of multiplications. So for brevity we do not mention M in what follows.

§ 2. Existence of the greatest P-system. Let P be a property on systems and let S be a system. Then a subsystem T of S is called the *greatest P-subsystem,* if it satisfies the following conditions:

(1) T is a P-system,

(2) if T' is a subsystem of S which satisfies P, then $T' \subset T$.

A property P is called *regular* (*preregular*), if for any system (at least one subsystem of which satisfies P), there exists its greatest P-subsystem.

Let $\{S_k: k \in K\}$ be a family of systems. We shall denote the free product of it by $S^* = \Pi^*\{S_k: k \in K\}$. Then we have the natural imbedding $i_k: S_k \to S^*$ by which we can regard S_k as a subsystem of S^*. Any quotient T of S^* under an onto homomorphism $h: S^* \to T$ is called a *semi-free product* if h sends each S_k into T in the one-to-one

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manner.

Remark. In this case we require only the one-to-one property between S_k and $S_k h$ for each $k \in K$. It does not imply that $S_j h$ and $S_k h$ are disjoint for $j \neq k$. They may meet.

Lemma. A property P on multiplicative systems is regular if and only if it is preregular and the empty system satisfies P.

Theorem 1. *A property P on multiplicative system is preregular (regular) if and only if it is semi-free product invariant (and the empty system satisfies P).*

Theorem 2. *If a property is both quotient and free product invariant (and moreover if it is satisfied by the empty system), then it is a preregular (regular) property.*

Example 1. A subsystem T of a system (S, M) is called total, if for any element $y \in T$, there exist $m_n \in M$ and $x_1, x_2, \cdots, x_n \in S$ such that $y=(x_1, x_2, \cdots, x_n)m_n$. Then for any system there exists its greatest total subsystem, which may be reduced to the empty subsystem.

Theorem 3. *For any property P, there exists a (pre)regular property $P^{\#}$ which satisfies the following conditions:*

(1) *$P^{\#}$ is (pre)regular.*

(2) *Any P-system is also a $P^{\#}$-system.*

(3) *If P' is such a (pre)regular property that any P-system is a P'-system, then any $P^{\#}$-system is also a P'-system.*

Further, such $P^{\#}$ is unique; in the sense that, if P'' is also such a (pre)regular property that satisfies the conditions then a system is a $P^{\#}$-system if and only if it is a P''-system.

§3. Amalgamated product. Let $\{S_k: k \in K\}$ be a family of systems, let A be a system and let $i_k: A \to S_k$ be an into isomorphism called an injection. We shall call the family $\{A, S_k, i_k: k \in K\}$ an *amalgamated family*. Form the free product $S^* = \Pi^*\{S_k: k \in K\}$ and consider the least congruence $\mathfrak{R}$ on S^* containing all pairs (xi_j, xi_k), $x \in A$, $j, k \in K$. Then we call the quotient system $T = S^*/\mathfrak{R}$ the *amalgamated product* of $\{A, S_k, i_k: k \in K\}$. Here A and i_k's are called the amalgam and the amalgam injection. Now we have the natural homomorphism $p_k: S_k \to T$ and $i: A \to T$ such that $i = i_k p_k$ for all $k \in K$. It is to be noted that the amalgamated product depends not only on its amalgam but also on its amalgam injections.

Theorem 4. *The above defined $p_k: S_k \to T$ is an into isomorphism for all $k \in K$, and so is $i: A \to T$.*

Any quotient of the amalgamated product of an amalgamated family $\{A, S_k, i_k: k \in K\}$ is called a *semi-amalgamated product*, if it is at the same time a semi-free product of $\{S_k: k \in K\}$.

If an amalgam is reduced to the empty system, then the notion of (semi-)amalgamated product coincides with that of (semi-)free product. A property P on systems is called *(semi-)amalgamated product invariant*, if the (any) (semi-)amalgamated product of any family of P-systems with non-empty amalgam, which is not necessarily a P-system, always satisfies the property P.

Let $\{S_k: k\in K\}$ be a family of subsystems of a system S. Let $\vee\{S_k: k\in K\}$ be the smallest subsystem of S containing all S_k's. Then there exists a unique homomorphism onto $\vee\{S_k: k\in K\}$ from the amalgamated product of the amalgamated family $\{A, S_k, i_k: k\in K\}$, where $A=\bigcap\{S_j: j\in K\}$ and i_k is the inclusion mapping, which sends every image $S_k p_k$ onto the subsystem S_k of $\vee\{S_k: k\in K\}$ in the one-to-one manner.

§4. Existence of the maximal P-systems. A property P on systems is called *semi-regular* (*-preregular*), if for any system S (at least one subsystem of which satisfies P) there exists a disjoint family of subsystems of S, $\{S_k: k\in K\}$, subject to the following conditions:

(1) S_k is a P-system for all $k\in K$.

(2) If T is a non-empty subsystem of S and if T is also a P-system, then there exists a unique $k\in K$ such that T is contained in S_k.

Theorem 5. *A property on multiplicative systems is a semi-preregular (-regular) property if and only if it is semi-amalgamated product invariant (and it is satisfied by the empty system).*

Theorem 6. *If a property on multiplicative systems is both quotient invariant and amalgamated product invariant, (and it is satisfied by the empty system), then it is a semi-preregular (-regular) property.*

Example 2. Let S be a multiplicative system with a single binary multiplication. Then S is called weakly left (right) simple, if there is no pair of disjoint proper left (right) ideals of S whose union is S. Then to be weakly left (right) simple is a semiregular property, i.e. for any given system with a binary multiplication, there exists a family of maximal weakly left simple subsystems which are mutually disjoint.

There are several examples more or less similar to the above example. They will be concerned in [2] in detail.

§5. Duality. As we have seen so far there are some nice contrasts between the notions in this paper and those in the previous one [1]. One theorem may be obtained just by changing all the terminologies to the corresponding dual ones. The following list shows the correspondence in this sense:

subsystem	quotient
one-to-one into mapping	onto mapping
isomorphism	isomorphism
into isomorphism	onto isomorphism
the trivial system	the empty system
(semi-)direct product	(semi-)free product
(semi-)spined product	(semi-)amalgamated product
(semi-)normal property	(semi-)regular property
(semi-)prenormal property	(semi-)preregular property
the greatest P-quotient	the greatest P-subsystem
inverse limit	direct limit

Here are some other examples of dualities in our sense:

In this section in what follows, whenever any notion of direct product, inverse limit, normal property, etc., is concerned, the empty set should not be considered as a system.

A system S is called *semi-free (-direct) product irreducible*, if for any family of systems, $\{S_k: k\in K\}$, such that S is a semi-free (-direct) product of $\{S_k: k\in K\}$, S is naturally isomorphic to S_k for at least one suffix $k\in K$.

Theorem 7. *A system is a semi-free (-direct) product of a family of semi-free (-direct) product irreducible systems.*

It is to be noted that a system is semi-free product irreducible if and only if it is generated by a single element.

The notion of direct (inverse) limit can be introduced in the usual way.

Theorem 8. *A semi-free (-direct) product of a family $\{S_j: j\in J\}$ is a direct (an inverse) limit of some family of systems, $\{T_k: k\in K\}$, where K is the directed set consisting of all non-empty finite subsets of J and T_k can be chosen as a semi-free (-direct) product of $\{S_j: j\in k\}$.*

Theorem 9. *A property P is semi-[pre]regular (normal), if and only if it satisfies the following conditions:*

(1) *For any two P-systems, any semi-free (-direct) product of them satisfies P.*

(2) *For any directed family of systems, all of which satisfy P, its direct (inverse) limit also satisfies P.*

[(3) *The empty (trivial) system satisfies P.*]

It is to be noted that the above condition (2) should be understood that it is satisfied automatically if the inverse limit vanishes.

To apply Theorem 9 upon concrete examples the following less general form may be preferable.

Theorem 10. *Let P be a property satisfying the condition[s] (2) [and (3)] as well as the following condition:*

(1′) *The free (direct) product of any two P-systems satisfies P. Then P is a semi-[pre]regular (normal) property.*

References

[1] Naoki Kimura: On some existence theorems on multiplicative systems. I. Greatest quotient, Proc. Japan Acad., **34**, 305 (1958).
[2] ——: On multiplicative systems (II) (unpublished).