72. On Some Existence Theorems on Multiplicative Systems. I. Greatest Quotient[1)]

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§1. Introduction. In this paper we shall give the necessary and sufficient condition for a property P on multiplicative systems in order that for any system there exists its greatest P-quotient.

In the subsequent paper we shall study the similar problems for the existence of maximal P-subsystems.

We shall state here only definitions and main theorems, proofs of which will be omitted. We shall give the details in [1].

§2. Main theorems. A pair (S, M) is called a *multiplicative system* or simply *system*, if

(1) S is a set, and

(2) $M=\bigcup_{n=1}^{\infty}M_n$, M_n's are disjoint, and element of M_n is a mapping $S^n\rightarrow S$, and is called an *n-ary multiplication*.

In this paper we assume, in addition, S is not empty; even though later in the subsequent paper we will permit the null set as a system. Also M_n may be empty.

Let (S, M) and (S', M') be two systems and let $q: M\rightarrow M'$ be a one-to-one correspondence which sends M_n onto M'_n. A mapping $f: S\rightarrow S'$ is called a *homomorphism* if for any $m_n\in M$, and for any $a_1, a_2,\cdots, a_n\in S$, the equality $(a_1f, a_2f,\cdots, a_nf)\ (m_nq)=(a_1, a_2,\cdots, a_n)m_nf$ holds. A one-to-one onto homomorphism is an isomorphism.

Since q is a one-to-one correspondence, we can identify M' with M. In what follows we assume that all systems considered have the same set M of multiplications.

Now it is easy to define the notion of subsystems, quotients, natural homomorphisms, congruences, direct products, free products, free systems, etc.

Any one element system is isomorphic to any other, and it is called *trivial*.

Let $\{S_j: j\in J\}$ be a family of systems. Let p_j be the projection of the direct product $S=\Pi\{S_j: j\in J\}$ onto S_j. Then any subsystem S' of S such that $S'p_j=S_j$ for all $j\in J$ is called a *semi-direct product*[2)]

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1) This work was supported in part by the National Science Foundation, U. S. A. This paper is an abstract of [1], which will be published elsewhere. Also see [2, pp. 1–52].

2) A semi-direct product defined here is a synonym for what is also called a subdirect product.

of the family $\{S_j : j \in J\}$. Clearly a semi-direct product is not unique in general.

Let P be a property on systems. Any system satisfying P is called a *P-system*. It is now easy to define for P to be subsystem invariant, direct product invariant, etc. For example, if every subsystem of any P-system also satisfies P, then P is subsystem invariant.

Let also S be a system. Let $\mathfrak{R}$ be a congruence on S such that;

(1) $S/\mathfrak{R}$ satisfies P,

(2) if $\mathfrak{R}'$ is a congruence on S such that $S/\mathfrak{R}'$ satisfies P, then $\mathfrak{R} \subset \mathfrak{R}'$.

Then we call $S/\mathfrak{R}$ the greatest P-quotient of S.

A property P is called *normal* (*prenormal*), if for any system (at least one quotient of which satisfies P) there exists its greatest P-quotient.

Theorem 1. *A property P is prenormal if and only if it is semi-direct product invariant.*[3]

Lemma. *A property P is normal if and only if it is prenormal and the trivial system satisfies P.*

Theorem 2. *A property P is normal if and only if it is semi-direct product invariant and the trivial system satisfies P.*[4]

Theorem 3. *If a property is subsystem invariant and direct product invariant (and also it is satisfied by the trivial system), then it is a prenormal (normal) property.*

Theorem 4. *For any property P, there exists a (pre)normal property P^* which satisfies the following conditions:*

(1) *P^* is (pre)normal.*

(2) *Any P-system is also a P^*-system.*

(3) *If P' is such a (pre)normal property that any P-system is a P'-system, then any P^*-system is also a P'-system.*

Further, such P^* is unique; in the sense that, if P'' is also such a (pre)normal property that satisfies the condition, then a system is a P^*-system if and only if it is a P''-system.

§3. Identities and implications. Let X be a set consisting of a countable number of distinct elements. Let F be the free system generated by X. Any element of F is called a word. Any pair of words, say (v, w), is called an *identity*. A set of identity $\{(v_j, w_j), j \in J; (v, w)\}$ is called an *implication*.

A system S is said to satisfy an identity (v, w) if for any homomorphism $h : F \to S$ always $vh = wh$ in S. Also S is said to satisfy an implication $\{(v_j, w_j), j \in J; (v, w)\}$ if for any homomorphism $h : F \to S$ such that $v_j h = w_j h$ for all $j \in J$ implies $vh = wh$.

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3) 4) This theorem has also been found by E. J. Tully.

An identity can be considered as an implication, but the converse is not true.

In multiplicative systems with a single binary multiplication, an associativity, a commutativity, an idempotency, etc., can be considered as identities, while a left cancellation law can be regarded as an implication.

A set of identities or implications can be considered as a property on systems.

As a characterization of properties defined by sets of identities, we have the following theorem by P. Hall. But any characterization theorem for properties defined by sets of implications is not yet known.

Theorem 5 (P. Hall). *A property is subsystem, direct product and quotient invariant if and only if it is a set of identities.*

Remark. This theorem was first obtained by P. Hall, but it has not been published.

Theorem 6. *Any set of implications (identities) is a normal property.*

§4. Examples. As applications, we shall consider here only systems with a single binary multiplication. As usual AB stands for the set of all products of $a \in A$ and $b \in B$. Even we shall give here only three examples of a little different sorts, we shall give several applications in [1].

Theorem 7. *Let S be any multiplicative system. Then there exists a disjoint family of left (right) ideals of S, $\{L_j : j \in J\}$, such that,*

(1) $S = \bigcup \{L_j : j \in J\}$,

(2) *If $\{L'_k : k \in K\}$ is a disjoint family of left (right) ideals of S such that $S = \bigcup \{L'_k : k \in K\}$, then for any $j \in J$ there exists $k \in K$ with $L_j \subset L'_k$* [2, p. 41].

Theorem 8. *Let S be an idempotent semigroup. Then there exists a disjoint family of rectangular subsemigroups $\{S_j : j \in J\}$ with a semilattice J as its suffix set such that*

(1) $S = \bigcup \{S_j : j \in J\}$,

(2) $S_i S_j \subset S_{ij}$,

(3) *If $\{S'_k : k \in K\}$ is a disjoint family of subsemigroups of S with a semilattice K as its suffix set such that $S = \bigcup \{S'_k : k \in K\}$ and $S'_k S'_l \subset S'_{kl}$, then for any $j \in J$ there exists $k \in K$ such that $S_j \subset S'_k$* [2, p. 50].

The notion of tensor product can be introduced in the usual fashion on systems.

Theorem 9. *Let S be a system. Then there exists a unique congruence $\mathfrak{R}$ on S such that*

(1) *$S/\mathfrak{R}$ is idempotent, and*

(2) *if $\mathfrak{R}'$ is such a congruence that $S/\mathfrak{R}'$ is idempotent then*

$\mathfrak{R} \subset \mathfrak{R}'$.

Furthermore $S/\mathfrak{R}$ is isomorphic with the tensor product of S with the trivial system.

§5. Maximal P-quotients. Let S and all S_j, $j \in J$, be systems and let $r_j : S_j \to S$ be an onto homomorphism for each $j \in J$. Let $\bar{S}$ be the direct product of S_j, $j \in J$, and let $p_j : \bar{S} \to S_j$ be the projection. For any element $z \in S$, the direct product $T_z = \Pi\{zr_j^{-1} : j \in J\}$ can be considered as a subset of $\bar{S}$. Define $T = \bigcup\{T_z : z \in S\}$. Then T is a subsystem of $\bar{S}$ which is called the *spined product* of $\{S_j : j \in J\}$. Here S, r_j are called the spine and the spine homomorphism, respectively. It is to be noted that the spined product depends not only on its spine but also on all of its spine homomorphisms. If the spine is trivial, then the spined product coincides with the usual direct product. A property P is called spined product invariant, if the spined product of any P-system with non-trivial spine (which need not be a P-system) is also a P-system. A semi-spined product is a subsystem of the spined product which is also a semi-direct product.

Let $\mathfrak{R}_1$ and $\mathfrak{R}_2$ be congruences on a system S. Let $\mathfrak{R}_1 \vee \mathfrak{R}_2$ be the least congruence containing $\mathfrak{R}_1$ and $\mathfrak{R}_2$. Then there exist the natural homomorphisms $p_i : S/(\mathfrak{R}_1 \frown \mathfrak{R}_2) \to S/\mathfrak{R}_i$, $q_i : S/\mathfrak{R}_i \to S/(\mathfrak{R}_1 \vee \mathfrak{R}_2)$ such that $p_1q_1 = p_2q_2$. Then $S/(\mathfrak{R}_1 \frown \mathfrak{R}_2)$ can be imbedded in the spined product T of $S/\mathfrak{R}_1$ and $S/\mathfrak{R}_2$ with $S/(\mathfrak{R}_1 \vee \mathfrak{R}_2)$ and q_i as its spine and the spine homomorphism. This imbedding is an onto isomorphism if and only if $\mathfrak{R}_1 \circ \mathfrak{R}_2 = \mathfrak{R}_1 \vee \mathfrak{R}_2$, where $\circ$ denotes the multiplication in the semi-group of all relations on S.

This can be generalized as follows.

Let $\{\mathfrak{R}_j : j \in J\}$ be congruences on a system S. Then $S/\bigcap\{\mathfrak{R}_j : j \in J\}$ is naturally isomorphic with a semi-spined product of $\{S/\mathfrak{R}_j : j \in J\}$ with $S/\vee\{\mathfrak{R}_j : j \in J\}$ as its spine and the natural homomorphisms as its spine homomorphisms.

A property P is called *semi-(pre)normal*, if and only if for any system S (at least one quotient of which satisfies P) there exists a family of congruences on S, $\{\mathfrak{R}_j : j \in J\}$, satisfying

(1) $S/\mathfrak{R}_j$ is a P-system,

(2) if $\mathfrak{R} \neq S \times S$ and if $S/\mathfrak{R}$ is a P-system, then there exists a unique $j \in J$ such that $\mathfrak{R}_j \subset \mathfrak{R}$.

Theorem 10. *A property on multiplicative systems is semi-prenormal (semi-normal) if and only if it is a semi-spined product invariant property (such that it is satisfied by the trivial system).*

§6. Supplement. So far in this paper we have considered properties on general multiplicative systems. However for the application, the study of more restricted system is preferable. If we replace the term "multiplicative system" by "Q-system", in the definition of normality, etc., then we have normality, etc., on Q-systems.

Theorem 11. *Let Q be a quotient invariant property. Then any (pre)normal property P on multiplicative systems is also a (pre)normal property on Q-systems. In other words, if S is a Q-system (at least one quotient of which satisfies P), where Q is a quotient invariant property, then there exists a congruence $\mathfrak{R}$ on S such that*

(1) *$S/\mathfrak{R}$ is a Q-system as well as a P-system,*

(2) *if $\mathfrak{R}'$ is such a congruence on S that $S/\mathfrak{R}'$ is both a Q-system and a P-system then $\mathfrak{R}\subset\mathfrak{R}'$.*

Theorem 12. *Let Q be a quotient invariant property. Then any semi-(pre)normal property P on multiplicative systems is also a semi-(pre)normal property on Q-systems. In other words, if S is a Q-system (at least one quotient of which satisfies P), then there exists a family of congruences on S, $\{\mathfrak{R}_j : j\in J\}$, such that*

(1) *$S/\mathfrak{R}_j$ is a Q-system as well as a P-system,*

(2) *if $\mathfrak{R}$ is such a congruence on S that $S/\mathfrak{R}$ is both a Q-system and a P-system and if $\mathfrak{R}\neq S\times S$ then there exists a unique $j\in J$ such that $\mathfrak{R}_j\subset\mathfrak{R}$.*

References

[1] Naoki Kimura: On multiplicative systems (I) (unpublished).
[2] ——: On semigroups, Dissertation, Tulane University, 1-133 (1957).