119. On Semi-continuity of Functionals. I

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1. *Introduction.* H. Nakano obtained the following decomposition theorem (cf. [3]). Let R be a *universally continuous semi-ordered linear space* (*a conditionally complete vector lattice* in Birkhoff's terminology [2]). A linear functional f on R is said to be *bounded* if $\sup_{|b|\leqq|a|}|f(b)|<+\infty$ for every $a\in R$.

A bounded linear functional f is decomposed into two linear functionals f_1, f_2 such that

$$f=f_1+f_2, \tag{1.1}$$

f_1 is a universally continuous linear functional[1] *and f_2 is orthogonal to every universally continuous linear functional.*

The structure of f_1 was fully discussed by Nakano [3]. But f_2 is not clear yet. On the other hand, he considered totally continuous spaces.[2]

Recently I. Amemiya conjectured that *if R is totally continuous, then f_2 is 0 on some complete semi-normal manifold of R.* (A semi-normal manifold (1-ideal in Birkhoff's terminology) $M\subset R$ is said to be *complete* if $|a|\frown|b|=0$, $a\in R$ (for every $b\in M$) implies $a=0$. In this case, it is proved that for any $a\in R$, there exist $a_\lambda\in M(\lambda\in\Lambda)$ with $a=\bigcup_{\lambda\in\Lambda}a_\lambda$ [3].)

The aim of this note is to prove Theorem 1 which deduces the Amemiya's conjecture as a special case. In the sequel, we will use the notations and terminologies in [3].

2. *Definitions and lemmas.* According to Nakano [3], R is said to be *totally continuous* if *for any double sequence of projectors* $[p_{i,j}]\uparrow_{j=1}^{\infty}[p]$, *there exists a sequence of projectors* $[p_k]\uparrow_{k=1}^{\infty}[p]$ *and integers* $l(i,k)$ $(i,k=1,2,\cdots)$ *with* $[p_k]\leqq[p_{i,l(i,k)}]$.

I. Amemiya proved that if the hypothesis of continuum is satisfied, then a universally continuous and totally continuous semi-ordered linear space is *super-universally continuous*, that is, for any *system* $a_\lambda\geqq0$ $(\lambda\in\Lambda)$, there exist countable $\lambda_i\in\Lambda$, a_{λ_i} $(i=1,2,\cdots)$ with $\bigcap_{\lambda\in\Lambda}a_\lambda=\bigcap_{i=1}^{\infty}a_{\lambda_i}$ [1]. If R is super-universally continuous and totally continuous, it is proved

1) $a_\lambda\downarrow_{\lambda\in\Lambda}0$ implies $\inf_{\lambda\in\Lambda}|f(a_\lambda)|=0$.

2) The space of measurable functions on measure space which is σ-finite is an example of totally continuous space.

that, 1st category set in the proper space $\mathfrak{E}$ of R is always nowhere dense, i.e. countable meet of open dense sets contains an open dense set in $\mathfrak{E}$ (cf. [3]).

We suppose that any functional in this paper is finite real-valued.

A functional m on R is said to be *monotone* if

(2.1) $$a, b \in R,\ |a| \leqq |b| \text{ implies } m(a) \leqq m(b).$$

We shall use the notation $a_\lambda \uparrow_{\lambda\in\Lambda} a$ (or $a_\lambda \downarrow_{\lambda\in\Lambda} a$) to mean $a = \bigcup_{\lambda\in\Lambda} a_\lambda$ (or $a = \bigcap_{\lambda\in\Lambda} a_\lambda$) and for all $\lambda_1, \lambda_2 \in \Lambda$ there exists $\lambda \in \Lambda$ with $a_\lambda \geqq a_{\lambda_1} \smile a_{\lambda_2}$ (or $a_\lambda \leqq a_{\lambda_1} \frown a_{\lambda_2}$).

A monotone functional m is said to be *semi-continuous* if

(2.2) $$0 \leqq a_\lambda \uparrow_{\lambda\in\Lambda} a \text{ implies } m(a) = \sup_{\lambda\in\Lambda} m(a_\lambda).$$

m is said to be *additive* if

(2.3) $$a \frown b = 0 \text{ implies } m(a+b) = m(a) + m(b).$$

m is said to be *coefficient-continuous* if

(2.4) for any sequence of numbers $0 \leqq \lambda_i \uparrow_{i=1}^{\infty} \lambda$ and for any $a \in R$, we have

$$m(\lambda a) = \sup_i m(\lambda_i a).$$

It is clear that (2.2) implies (2.4).

Lemma 1. *A monotone and coefficient-continuous functional m on a universally continuous semi-ordered linear space R is semi-continuous if and only if, for any $0 \leqq p \in R$, $[p_\lambda] \uparrow_{\lambda\in\Lambda} [p]$ implies*

$$m(p) = \sup_{\lambda\in\Lambda} m([p_\lambda]p).$$

Proof. We shall only prove the sufficiency. Let $0 \leqq a_\lambda \uparrow_{\lambda\in\Lambda} a$.

Putting $(a_\lambda - (1-\varepsilon)a)^+ = p_\lambda$ $(\varepsilon > 0)$, we have

(2.5) $$a_\lambda \geqq [p_\lambda]a_\lambda \geqq (1-\varepsilon)[p_\lambda]a.$$

Since $p_\lambda \uparrow_{\lambda\in\Lambda} \varepsilon a$, we have

(2.6) $$[p_\lambda] \uparrow_{\lambda\in\Lambda} [a].$$

Because m is monotone, by (2.5) and (2.6), we have

$$\sup_{\lambda\in\Lambda} m(a_\lambda) \geqq m((1-\varepsilon)a).$$

Since m is coefficient-continuous, we have

$$\sup_{\lambda\in\Lambda} m(a_\lambda) = m(a).$$

Remark. In the above lemma, if m is furthermore an additive functional, we can replace the statement "$[p_\lambda] \uparrow_{\lambda\in\Lambda} [p]$ implies $m(p) = \sup_{\lambda\in\Lambda} m([p_\lambda]p)$" by "$[p_\lambda] \downarrow_{\lambda\in\Lambda} 0$ implies $\inf_{\lambda\in\Lambda} m([p_\lambda]p) = 0$".

Lemma 2. *If R is super-universally continuous, a monotone functional m on R is semi-continuous if and only if for any $0 \leqq p \in R$, $[p_i] \uparrow_{i=1}^{\infty} [p]$ implies $m(p) = \sup_i m([p_i]p)$.*

Proof of this lemma is clear from Lemma 1.

3. *Theorem* 1. *Let R be super-universally continuous and totally continuous, and let m be a functional on R which is additive, mono-*

tone, and coefficient-continuous. Then m is semi-continuous on some complete semi-normal manifold of R.

Proof. For any $a \in R$, we put

$$B_a = \{b : |b| \leqq \lambda_b |a| \text{ for some } \lambda_b > 0\}.$$

When m is semi-continuous on B_a, we call a a *semi-continuous element* by m. The totality of semi-continuous elements is denoted by R_s.

(1) If $a, b \in R_s$, then $a+b$, $\lambda a \in R_s$.

(2) If $|a| \geqq |b|$, $a \in R_s$, $b \in R$, then $b \in R_s$.

(3) m is semi-continuous on R_s.

(2) and (3) are clear, therefore we shall only prove (1).

For any number $\lambda \neq 0$, we have $B_{\lambda a} = B_a$, and so if a is a semi-continuous element by m, then λa is semi-continuous element for all numbers λ.

Let c be an element such that

$$B_{a+b} \ni c \geqq 0,\ [c_\lambda] \uparrow_{\lambda \in \Lambda} [c] \text{ where } c_\lambda \in B_{a+b} (\lambda \in \Lambda).$$

Whithout loss of generality, we can suppose $a \geqq 0$, $b \geqq 0$ and $a+b \geqq c \geqq 0$. We put $c_1 = [(a-b)^+]c$ and $c_2 = c - [(a-b)^+]c$.

Since $[(a-b)^+]a \geqq [(a-b)^+]b$

and

$a - [(a-b)^+]a \leqq b - [(a-b)^+]b$

imply

$[(a-b)^+]c \leqq [(a-b)^+](a+b) \leqq [(a-b)^+]2a \leqq 2a$

and

$c - [(a-b)^+]c \leqq a+b-[(a-b)^+](a+b) \leqq 2(b-[(a-b)^+]b) \leqq 2b$

respectively,

we see thus;

$$0 \leqq c_1 \leqq 2a \quad \text{and} \quad 0 \leqq c_2 \leqq 2b,$$
$$c_1 + c_2 = c \quad \text{and} \quad c_1 \frown c_2 = 0,$$

therefore, $c_1 \in B_a$ and $c_2 \in B_b$.

Since m is semi-continuous on B_a and B_b, and additive on R, we see that

$$m(c) \geqq \sup_{\lambda \in \Lambda} m([c_\lambda]c) = \sup_{\lambda \in \Lambda} (m([c_\lambda]c_1) + m([c_\lambda]c_2))$$
$$= \sup_{\lambda \in \Lambda} m([c_\lambda]c_1) + \sup_{\lambda \in \Lambda} m([c_\lambda]c_2) = m(c_1) + m(c_2) = m(c).$$

By Lemma 1, m is semi-continuous on B_{a+b}, this proves the first part of (1).

(1), (2), (3) show that R_s is a semi-normal manifold of R.

We shall show that R_s is complete in R. For this purpose, we shall prove that, for any $a > 0$, $a \in R$ there exists an element b with $a \geqq b > 0$, $b \in R_s$.

For a positive number $\varepsilon > 0$, if we can find a sequence of projectors $[a]\geqq[p_i]$ $(i=1,2,\cdots)$ with $[p_i]\downarrow_{i=1}^{\infty}0$, and $\inf_i m([p_i]a)\geqq\varepsilon$, then $A_1=U_{[a]}-\bigcap_i U_{[p_i]}$ is open dense in $U_{[a]}$, because $U_{[p_i]}$ are open and closed. Furthermore, if we can find a sequence of projectors $[a]\geqq[q_i]\downarrow_{i=1}^{\infty}0$ $(i=1,2,\cdots)$ with

$$U_{[q_i]}\subset A_1\ (i=1,2,\cdots),\quad\text{and}\quad \inf_i m([q_i]a)\geqq\varepsilon,$$

then we see easily that $A_2=A_1-\bigcap_i U_{[q_i]}$ is open dense in $U_{[a]}$.

Since $U_{[p_i]}$, $U_{[q_j]}$ $(i,j=1,2,\cdots)$ are compact, we can find $[p_{i_0}]$ and $[q_{j_0}]$ such that

$$U_{[p_{i_0}]}\frown U_{[q_{j_0}]}=\phi,$$

but this is equivalent to

$$[p_{i_0}]\frown[q_{j_0}]=0.$$

If we proceed with this method, and we can find mutually orthogonal projectors $[a]\geqq[p'_i]$ $(i=1,2,\cdots,n)$ with

$$(4)\qquad m([p'_i]a)\geqq\varepsilon\quad (i=1,2,\cdots,n),$$

then

$$m(a)\geqq m(([p'_1]+\cdots+[p'_n])a)=\sum_{i=1}^{n}m([p'_i]a)\geqq n\varepsilon.$$

Because $m(a)<+\infty$, we can not find infinite numbers of mutually orthogonal projectors satisfying (4), i.e. we find finite numbers of open dense sets $A_1,A_2,\cdots,A_n\subset U_{[a]}$ such that

$$\bigcap_{i=1}^{n}A_i\supset U_{[p_j]},\ [p_j]\downarrow 0\ \text{implies}\ \inf_j m([p_j]a)\leqq\varepsilon.$$

For any k $(k=1,2,\cdots)$, we can find, therefore, an open dense set $B_k\subset U_{[a]}$ such that

$$B_k\supset U_{[p_i]}\ (i=1,2,\cdots)\ \text{and}\ [p_i]\downarrow_i 0\ \text{imply}\ \inf_i m([p_i]a)\leqq\frac{1}{k}.$$

Because R is super-universally and totally continuous, we find an open dense set B' in $U_{[a]}$ with $B'\subset\bigcap_{k=1}^{\infty}B_k$.

For any $[p_i]$ $(i=1,2,\cdots)$ such that $B'\supset U_{[p_i]}$, $[p_i]\downarrow_{i=1}^{\infty}0$, we have $\inf_i m([p_i]a)=0$.

By the same method, for any $n=2,\cdots$, we can find an open dense set B'_n in $U_{[a]}$ such that

$$B'_n\supset U_{[p_i]}\ \text{and}\ [p_i]\downarrow_{i=1}^{\infty}0\ \text{imply}\ \inf_i m(n[p_i]a)=0.$$

Because R is super-universally and totally continuous, $\bigcap_n B'_n$ contains an open dense set B in $U_{[a]}$; therefore we can find a projector $[b']\neq 0$ with $U_{[b']}\subset B$. Let $[p_i]\downarrow_{i=1}^{\infty}0$, then $\inf_i m(n[p_i][b']a)=0$. Hence, putting $b=[b']a$, we see easily $0<b\leqq a$ and $b\in R_s$ by the remark of Lemma 1 and Lemma 2. Hence, R_s is a complete semi-normal manifold of R. From the definition of R_s, it is clear that m is semi-continuous on R_s. This proves the theorem.

4. *Theorem* 2 (*Amemiya's conjecture*). *If a bounded linear functional is orthogonal to every universally continuous linear functional, then this functional is* 0 *on some complete semi-normal manifold. Any bounded linear functional coincides with a universally continuous linear functional on some complete semi-normal manifold.*

Proof. Any bounded linear functional f can be written by the difference of two positive linear functionals. We prove Theorem 2 only in the case that f is positive (i.e. $a \geqq 0$ implies $f(a) \geqq 0$). For, if the sets $A, B \subset R$ are complete semi-normal manifolds, then $A \frown B$ is also. The functional m defined by $m(a)=f(|a|)$, $a \in R$ is monotone, coefficient-continuous, and additive. By Theorem 1, we see easily the validity of Theorem 2.

References

[1] I. Amemiya: On totally continuous space (in Japanese), Functional Analysis, **2-6** (1952).

[2] G. Birkhoff: Lattice Theory, Amer. Math. Col. (1948).

[3] H. Nakano: Modulared Semi-ordered Linear Space, Tokyo Math. Book Series (1950).