118. *Notes on Lattices*

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(Comm. by K. KUNUGI, M.J.A., Oct. 13, 1958)

Let L be a lattice with an inclusion relation $\leqq$, meet $a\frown b$ and join $a\smile b$. L. M. Blumenthal and D. O. Ellis [2] showed that the follwing three relations (G), (G*) and (G**) are equivalent in modular lattices, and that they are also equivalent to metric betweeness for normed lattices.

(G) $\quad (a\frown c)\smile(b\frown c)=c=(a\smile c)\frown(b\smile c)$

(G*) $\quad (a\frown c)\smile(b\frown c)=c=c\smile(a\frown b)$

(G**) $\quad (a\smile c)\frown(b\smile c)=c=c\frown(a\smile b)$

Recently, Y. Matsushima [3] introduced for any lattice L three kinds of sets in L as follows:*)

$$J(a, b)=\{x\mid x=(a\frown x)\smile(b\frown x)\}$$
$$CJ(a, b)=\{x\mid x=(a\smile x)\frown(b\smile x)\}$$
$$B(a, b)=J(a, b)\wedge CJ(a, b).$$

He gave among others a characterization of distributive lattices by using $B(a, b)$, and a characterization of modular lattices by using $B(a, b)$ and $B^*(a, b)$ in [3, 4].

In this note we give some characterizations of modular lattices by $J(a, b)$ and $CJ(a, b)$, which also imply that (G), (G*) and (G**) are equivalent only in modular lattices. We also give two characterizations of distributive lattices by using $J(a, b)$ and $CJ(a, b)$ respectively, each of which is the dual of the other.

LEMMA 1. *If (a, b) is a modular pair* [1, p. 100], *then $[a\frown b, b]$ is contained in $CJ(a, b)$.*

PROOF. Choose x from $[a\frown b, b]$; then $x\leqq b$ and $(x\smile a)\frown(x\smile b)=(x\smile a)\frown b=x\smile(a\frown b)$ since (a, b) is a modular pair. While $a\frown b\leqq x$, we have $(x\smile a)\frown(x\smile b)=x$. This shows that $[a\frown b, b]\subset CJ(a, b)$.

LEMMA 2. *If $[a\frown b, b]$ is contained in $CJ(a, b)$, then (a, b) is a modular pair.*

PROOF. Let $x\leqq b$, and consider $x\smile(a\frown b)$. Then $a\frown b\leqq x\smile(a\frown b)\leqq b$ and hence by assumption $x\smile(a\frown b)\in CJ(a, b)$. Hence $x\smile(a\frown b)=(x\smile(a\frown b)\smile a)\frown(x\smile(a\frown b)\smile b)=(x\smile a)\frown(x\smile b)=(x\smile a)\frown b$. This shows that (a, b) is a modular pair.

LEMMA 2′. *If $[b, a\smile b]\subset J(a, b)$ for any two elements a and b, then L is a modular lattice.*

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*) We denote the set-theoretical inclusion and intersection by $\subset$ and $\wedge$. We also use $[a)$, $(a]$ and $[a, b]$ for $\{x\mid a\leqq x\}$, $\{x\mid x\leqq a\}$ and $\{x\mid a\leqq x\leqq b\}$ respectively.

PROOF. For $x \leqq z$, and any y, we consider $(x \smile y) \frown z$. Then $x \leqq (x \smile y) \frown z \leqq y \smile x$, and hence we have $(x \smile y) \frown z \in J(y, x)$ by assumption. Consequently $(x \smile y) \frown z = ((x \smile y) \frown z \frown y) \smile ((x \smile y) \frown z \frown x) = (z \frown y) \smile (z \frown x) = x \smile (z \frown y)$.

LEMMA 3. *In a modular lattice L, we have $J(a, b) \wedge [a \frown b] \subset CJ(a, b)$ for any two elements a and b.*

PROOF. Let c be in $J(a, b) \wedge [a \frown b)$. Then $(a \smile c) \frown (c \smile b) = c \smile (a \frown (c \smile b))$ by modularity, and $a \frown (c \smile b) = a \frown ((a \frown c) \smile (c \frown b) \smile b)$ since $c \in J(a, b)$, and hence we have $a \frown (c \smile b) = a \frown ((a \frown c) \smile b) = (a \frown c) \smile (b \frown a)$ by using modularity again. Consequently $(a \smile c) \frown (c \smile b) = c \smile (a \frown c) \smile (b \frown a) = c \smile (b \frown a) = c$ since $a \frown b \leqq c$. This shows that c is in $CJ(a, b)$, and $J(a, b) \wedge [a \frown b) \subset CJ(a, b)$.

LEMMA 3′. *In a modular lattice, we have $CJ(a, b) \wedge (a \smile b] \subset J(a, b)$ for any two elements a and b.*

PROOF. If c is in $CJ(a, b) \wedge (a \smile b]$, we have $c = (a \smile c) \frown (c \smile b)$ and $c \frown b = (a \smile c) \frown (c \smile b) \frown b = (a \smile c) \frown b$. Using modularity and this relation, we have $(a \frown c) \smile (c \frown b) = (a \smile (c \frown b)) \frown c = (a \smile ((a \smile c) \frown b)) \frown c = (a \smile b) \frown (a \smile c) \frown c = (a \smile b) \frown c$. Since $c \leqq a \smile b$, we have $(a \frown c) \smile (c \frown b) = c$. This shows that $c \in J(a, b)$ and $CJ(a, b) \wedge (a \smile b] \subset J(a, b)$.

REMARKS. Let us consider a lattice $P = \{p, q, r, s \text{ and } d\}$ such that $p < q < r < s$, $p < d < s$, $q \frown d = r \frown d = p$, and $q \smile d = r \smile d = s$.

(1) $[r, d \smile r] \subset J(d, r)$, but (d, r) is not a modular pair, since $q \smile (d \frown r) \neq (q \smile d) \frown r$.

(2) (r, d) is a modular pair, and $q \in J(r, d) \wedge [r \frown d)$ but $q \notin CJ(r, d)$.

(3) (q, d) is a modular pair, and $r \in CJ(q, d) \wedge (q \smile d]$ but r is not in $J(q, d)$.

THEOREM 1. *A necessary and sufficient condition for L to be a a modular lattice is that $J(a, b) \wedge [a \frown b) \subset CJ(a, b)$ for every pair a and b.*

PROOF. If L is a modular lattice, then $J(a, b) \wedge [a \frown b) \subset CJ(a, b)$ by Lemma 3. If $J(a, b) \wedge [a \frown b) \subset CJ(a, b)$, we have $[a \frown b, b] \subset CJ(a, b)$ since $[a \frown b, b] \subset J(a, b)$ in any lattice. Hence L is a modular lattice by Lemma 2.

THEOREM 1′. *A necessary and sufficient condition for L to be modular lattice is that $CJ(a, b) \wedge (a \smile b] \subset J(a, b)$ for every pair a and b.*

PROOF. If L is modular, we have $CJ(a, b) \wedge (a \smile b] \subset J(a, b)$ by Lemma 3′. If $CJ(a, b) \wedge (a \smile b] \subset J(a, b)$, we have $[b, a \smile b] \subset J(a, b)$ since $[b, a \smile b] \subset CJ(a, b)$ in any lattice. Consequently L is a modular lattice by Lemma 2′.

COROLLARY. (G), (G*) *and* (G**) *are equivalent if and only if L is a modular lattice.*

PROOF. In any lattice, if c satisfies (G), then c is in $J(a, b) \wedge$

$CJ(a, b)$. Hence $a \frown b \leqq c \leqq a \smile b$ [3, Theorem 1] and c satisfies (G^*) and (G^{**}). Let L be a modular lattice. If c satisfies (G^*), then c is in $J(a, b) \wedge [a \frown b) \subset CJ(a, b)$ and c satisfies (G); if c satisfies (G^{**}), then c is in $CJ(a, b) \wedge (a \smile b] \subset J(a, b)$, and c satisfies (G). Thus we have shown that (G), (G^*), (G^{**}) are equivalent in a modular lattice. Conversely, let (G) and (G^*) be equivalent in L. Then we have $J(a, b) \wedge [a \frown b) \subset CJ(a, b)$ for every pair a and b, and hence by Theorem 1 L is a modular lattice. If (G) and (G^{**}) are equivalent in L, we have $CJ(a, b) \wedge (a \smile b] \subset J(a, b)$ for every pair a and b, and hence by Theorem 1$'$ L is a modular lattice. If (G^*) and (G^{**}) are equivalent in L, we have $J(a, b) \wedge [a \frown b) \subset CJ(a, b)$ and $CJ(a, b) \wedge (a \smile b] \subset J(a, b)$ for every pair a and b, and L is a modular lattice.

THEOREM 2. *A necessary and sufficient condition for L to be a distributive lattice is that $J(a, b)$ be an ideal for every pair a and b. In this case we have $J(a, b) = (a \smile b]$.*

PROOF. In any lattice, $J(a, b) \subset (a \smile b]$ [3, Theorem 1] and for any two elements x and y in $J(a, b)$, $x \smile y$ is also in $J(a, b)$ [3, p. 549]. Let L be a distributive lattice and $x \leqq t$, $t \in J(a, b)$. Then $(x \frown a) \smile (x \frown b) = x \frown (a \smile b) = x$, and $x \in J(a, b)$. This shows that $J(a, b)$ is an ideal. If $J(a, b)$ is an ideal, $J(a, b) = (a \smile b]$ since $J(a, b)$ contains $a \smile b$. Conversely, let $J(a, b)$ be an ideal. Then $J(a, b) = (a \smile b]$. For any elements x, a and b in L, we have $x \frown (a \smile b) \in J(a, b)$. Hence $x \frown (a \smile b) = (x \frown (a \smile b) \frown a) \smile (x \frown (a \smile b) \frown b) = (x \frown a) \smile (x \frown b)$. This means that L is a distributive lattice.

Dually we have the following

THEOREM 2$'$. *For any pair a and b, $CJ(a, b)$ is a dual ideal if and only if L is a distributive lattice. In this case we have $CJ(a, b) = [a \frown b)$.*

References

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