117. *On Ideals in Semiring*

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Recently, G. Thierrin [6, 7] has discussed some kinds of ideals in any ring and associative (i.e. semigroup). An associative means an algebraic system with an associative law, i.e. semigroup. This terminology was suggested by Dr. F. Klein in his letter (May 15, 1958) to the present writer. In his papers [1, 2], F. Klein has stated that the word is better than semigroup. L. Lesieur and R. Croisot [3–5] have developed a new unified theory of ideals in ring, associative and module. In this paper, we shall consider ideals in a semiring. For fundamental notions on a semiring and its related subjects, see H. S. Vandiver and M. W. Weaver [8].

An ideal P in a semiring is called *completely prime*, if $ab \in P$ implies $a \in P$ or $b \in P$.

An ideal M is called *completely semi-prime*, if $a^2 \in M$ implies $a \in M$.

Following G. Thierrin [6, 7], we shall define a *compressed ideal* in a semiring R. An ideal M is *compressed*, if and only if $a_1^2 a_2^2 \cdots a_n^2 \in M$ for any n implies $a_1 a_2 \cdots a_n \in M$.

Every completely prime ideal is compressed, and every compressed ideal is completely semi-prime.

Theorem 1. *If an ideal M is compressed, then $a_1 a_2 \cdots a_n \in M$ implies $a_1^{l_1} a_2^{l_2} \cdots a_n^{l_n} \in M$ for any positive integers $l_1, l_2, \cdots, l_n$, and $a_1^{l_1} a_2^{l_2} \cdots a_n^{l_n} \in M$ implies $a_1 a_2 \cdots a_n \in M$.*

Proof. $a_1 a_2 \cdots a_n \in M$ implies $a_1^{l_1} a_2 \cdots a_n \in M$. By the remark above, M is completely prime, and we have $a_2 \cdots a_n a_1^{l_1} \in M$.[1] By the same argument, we have $a_2^{l_2} a_3 \cdots a_1^{l_1} \in M$. Hence we have $a_3 \cdots a_n a_1^{l_1} a_2^{l_2} \in M$. This implies $a_1^{l_1} a_2^{l_2} \cdots a_n^{l_n} \in M$.

Conversely, let $a_1^{l_1} a_2^{l_2} \cdots a_n^{l_n} \in M$, then we have $a_1^l a_2^l \cdots a_n^l \in M$ for sufficiently large integer $n > 2$. Therefore, we have $a_1^{2(l-1)} a_2^{2(l-1)} \cdots a_n^{2(l-1)} \in M$. Since M is compressed, then we have $a_1^{l-1} a_2^{l-1} \cdots a_n^{l-1} \in M$. By repeating the same processes, we have $a_1 a_2 \cdots a_n \in M$.

Let M be an ideal in R. We shall call an element x T-element for M if it is $x = x_1 \cdot x_2 \cdots x_n$ such that $x_1^2 x_2^2 \cdots x_n^2 \in M$ for some n. Let us denote by $T^1(M)$ the set of all T-elements for M, and let $T_1(M)$ be the ideal generated by $T^1(M)$. By the induction, we shall $T_n(M)$ as follows: $T_n(M) = T_1(T_{n-1}(M))$ $(n > 1)$. Then each $T_n(M)$ is an ideal and $T_n(M) \subseteq T_{n+1}(M)$. The Thierrin radical $T^*(M)$ of M is

1) See K. Iséki: Ideals in semirings, Proc. Japan Acad., **34**, 29–31 (1958).

the set-sum of $T_n(M)$ $(n=1, 2, \cdots)$, i.e. $T^*(M)=\bigcup_{n=1}^{\infty} T_n(M)$. It is clear that an ideal of a semiring is compressed, if and only if it is the Thierrin radical of the ideal.

Theorem 2. *The Thierrin radical $T^*(M)$ of an ideal M is the intersection of all compressed ideal containing M.*

Proof. First we shall show that $T^*(M)$ is a compressed ideal containing M. Clearly, $T^*(M)\supset M$ and $T^*(M)$ is an ideal. To show that $T^*(M)$ is compressed, let $x_1^2x_2^2\cdots x_n^2\in T^*(M)$, then we have $x_1^2x_2^2\cdots x_n^2\in T_m(M)$ for some m. Hence, $x_1x_2\cdots x_n\in T_{m+1}(M)$. Therefore $x_1x_2\cdots x_n\in T^*(M)$.

Next, let N be a compressed ideal containing M. Then we shall show $T^*(M)\subset N$. Let $x\in T^1(M)$, then we have $x=x_1x_2\cdots x_n$ such that $x_1^2x_2^2\cdots x_n^2\in M$. Since $M\subset N$, and N is compressed, we have $x\in N$. Hence $T_1(M)\subset N$. By an easy induction, we have $T_n(M)\subset N$. Therefore we have $T^*(M)\subset N$, consequently, $T^*(M)$ is the intersection of all compressed ideals containing M.

A subset M of R is an m-system if and only if $a, b\in M$ imply that there is an element x of R such that $axb\in M$. We shall define the McCoy radical of an ideal in a semiring as follows. The McCoy radical of an ideal M in R is the set of all elements x such that every m-system which contains x contains an element of M.

Theorem 3. *The McCoy radical of an ideal M in a semiring R is contained in the Thierrin radical $T^*(M)$ of M.*

Proof. Let a be an element of the McCoy radical of M, then some power a^n of a is an element in M. Hence $a^n\in M\subset T^*(M)$. Since $T^*(M)$ is compressed, $a\in T^*(M)$.

A prime (completely prime) ideal P is a *minimal prime* (*completely prime*) *ideal belonging to the ideal* M if and only if $M\subset P$ and there is no prime (completely prime) ideal P' such that $M\subset P'\subset P$, $P'\neq P$.

Theorem 4. *Any minimal prime ideal P belonging to a compressed ideal M is a completely prime ideal belonging to M.*

Proof. Suppose that $R-P$ is not empty. Then $R-P$ is an m-system[2] and $R-P$ is a maximal m-system which does not meet M. Let $C(P)$ be the set of all elements of $a=x_1^{m_1}x_2^{m_2}\cdots x_k^{m_k}$, where $x_1x_2\cdots x_k\in R-P$, $m_1, m_2, \cdots, m_k$ are positive integers and $k=1, 2, \cdots$. Clearly $R-P\subset C(P)$. To prove that $C(P)$ is an m-system, let $b=y_1^{n_1}y_2^{n_2}\cdots y_l^{n_l}\in C(P)$ and $y_1, y_2, \cdots y_l\in R-P$. Then there is an element x of R such that $x_1x_2\cdots x\ y_1y_2\cdots y_n\in R-P$. Therefore $x_1^{m_1}x_2^{m_2}\cdots x_k^{m_k}x\ y_1^{n_1}y_2^{n_2}\cdots y_l^{n_l}\in C(P)$. This shows $axb\in C(P)$, and $C(P)$ is an m-system. We shall prove that $C(P)$ is an m-system. We shall prove that $C(P)$ does not

2) See K. Iséki: Loc. cit., p. 29.

meet M. Suppose that $M\frown C(P)\neq\phi$. Then there is an element c such $c\in M\frown C(P)$. Hence $C=Z_1^{t_1}Z_2^{t_2}\cdots Z_q^{t_q}$, and $Z_1Z_2\cdots Z_q\in R-P$.

Since M is compressed, we have $Z_1Z_2\cdots Z_q\in M$. Hence $M\frown(R-P)\neq\phi$. Therefore we have $R-P=C(P)$. Finally, to prove that P is a completely prime ideal, let $x^2\in P$. Suppose that $x\notin P$. Then $x\in R-P=C(P)$. Hence $x^2\in C(P)$. This shows $x^2\in P$. Therefore, P is completely prime.

We shall remark that the following fundamental theorems are obtained by the techniques of N. H. McCoy and G. Thierrin.

Theorem 5. *The Thierrin radical $T^*(M)$ of an ideal M is the intersection of all minimal completely prime ideals belonging to M.*

Theorem 6. *Any compressed ideal M is the intersection of all minimal completely prime ideals belonging to M.*

References

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