116. *Finite-to-one Closed Mappings and Dimension. I*[1)]

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The fundamental theorem of this note is as follows.

Theorem 1. *Let R and S be metric spaces and f a closed mapping (continuous transformation) of R onto S. If $f^{-1}(y)$ consists of exactly $k(<\infty)$ points for every point $y\in S$ and* $\dim R\leq 0$, *then we have* $\dim S\leq 0$.[2)]

As direct consequences of this theorem we get a large number of theorems of dimension theory for non-separable metric spaces, among which there is Morita-Katětov's fundamental theorem of dimension theory. This fact indicates the possibility of the development of dimension theory, other than Morita and Katětov's, for non-separable metric spaces based on Theorem 1. An analogue to Theorem 1 for the case when f is open will also be stated.

Lemma 1. *R is a metric space with* $\dim R\leq 0$, *if and only if R is a dense subset of an inverse limiting space of a sequence of discrete spaces.*

This is a trivial modification of Morita [2, Theorem 10.2] or of Katětov [1, Theorem 3.6]; its proof is included in that of Theorem 4 below.

Proof of Theorem 1. By Lemma 1 we can assume that R is a dense subset of $\lim R_i$ obtained from $\{R_i, f_{jk}: R_j\to R_k\ (j>k)\}$ with discrete spaces $R_i=\{p_{i\alpha};\ \alpha\in A_i\}$. We can assume that points of R_i are linearly-ordered such that for any $p_{i\alpha}, p_{i\beta}$ with $f_{ij}(p_{i\alpha})\neq f_{ij}(p_{i\beta})$, $i>j$, it holds that $p_{i\alpha}>p_{i\beta}$ if and only if $f_{ij}(p_{i\alpha})>f_{ij}(p_{i\beta})$. We introduce into points $(p_{1\alpha_1}, p_{2\alpha_2},\cdots)$ of $\lim R_i$ the lexicographic order with respect to the one of R_i just defined. Let $x_1(y),\cdots,x_k(y)\in R$ be the inverse image of $y\in S$ with $x_1(y)<\cdots<x_k(y)$ and then R is decomposed into mutually disjoint subsets $T_i=\{x_i(y);\ y\in S\}$, $i=1,\cdots,k$.

We shall show that every T_i is an F_σ. To do so it suffices to prove T_1 is an F_σ since the rest case is proved similarly. Let $r(y)$, $y\in S$, be the smallest integer such that $\pi_r(x_1(y)),\cdots,\pi_r(x_k(y))$ are mutually different points of R_r, where $\pi_r:\lim R_i\to R_r$ is the natural projection. Let $S_t=\{y;\ y\in S,\ r(y)\leq t\}$, $t=1,2,\cdots$, and $T_{jt}=T_j\cap f^{-1}(S_t)$ and then evidently i) $S=\bigcup_{t=1}^{\infty}S_t$, ii) $T_1=\bigcup_{t=1}^{\infty}T_{1t}$, iii) $T_{1t}\subset T_{1,t+1}$. The

1) The detail of the content of the present note will be published in another place.
2) dim = covering dimension.

family $\{f(V(p_{t\alpha})); V(p_{t\alpha})=\{x; x\in R, \pi_t(x)=p_{t\alpha}\}, \alpha\in A_t\}$ is a closed covering of S such that the sum of any subfamily is also closed. Let y be an arbitrary point in S_t and then it is not hard to see that $W=S-\bigcup\{f(V(p_{t\alpha})); \alpha\in A_t, y\notin f(V(p_{t\alpha}))\}$ is an open set of S which contains y and that $z\in S_t\cap W$ implies $\pi_t(x_j(y))=\pi_t(x_j(z))$ for $j=1,\cdots,k$. Therefore an open set $G_{ty}=\bigcup_{j=2}^{k}(f^{-1}(W)\cap V(\pi_t(x_j(y))))$ is unable to meet T_{1t}. Thus $F_t=R-\bigcup\{G_{ty}; y\in S_t\}$ is a closed set with $F_t\supset T_{1t}$ and $F_t\cap(\bigcup_{j=2}^{k}T_{jt})=\phi$. Since $H_j=\bigcap_{t=j}^{\infty}F_t$ is a closed set with $H_j\supset T_{1j}$ and $H_j\cap(\bigcup_{i=2}^{k}T_i)=\phi$, $T_1=\bigcup_{j=1}^{\infty}H_j$ and T_1 is an F_σ. Since $f|H_j$ is a homeomorphism, $\dim f(H_j)\leq 0$. Moreover $f(H_j)$ is closed in S and $S=\bigcup_{j=1}^{\infty}f(H_j)$ and hence $\dim S\leq 0$ by the sum theorem.

We enumerate consequences of this theorem with sketch of proofs or without proofs.

Theorem 2. *Let R and S be metric spaces with* $\dim R\leq 0$ *and f a closed mapping of R onto S such that $f^{-1}(y)$ is a finite set at every point $y\in S$. Then for any finite m, we have* $\dim\{y; |f^{-1}(y)|=m\}\leq 0$.

Theorem 3. *Let R and S be metric spaces with* $\dim R\leq 0$ *and f a closed finite-to-one mapping of R onto S. Then* $\dim S\leq|\{i; \{y; |f^{-1}(y)|=i\}\neq\phi\}|-1$.

Theorem 4 (Morita [3, Theorem 4]). *Let R be a metric space. Then* $\dim R\leq n(<\infty)$ *if and only if R is the image of a metric space R_0 with* $\dim R_0\leq 0$ *under a closed mapping f such that $f^{-1}(y)$ consists of at most $n+1$ points for every point $y\in R$.*

Proof. The sufficiency is evident from Theorem 3, and hence we show that the condition is necessary. Let $\mathfrak{U}_1=\{U_\alpha; \alpha\in A_1\}$ be a locally finite open covering of R of order $\leq n+1$ such that the diameter of each $U_\alpha<1$. Then there exist a closed covering $\mathfrak{F}_1=\{F_\alpha; \alpha\in A_1\}$ and an open covering $\mathfrak{V}_1=\{V_\alpha; \alpha\in A_1\}$ such that $U_\alpha\supset F_\alpha\supset V_\alpha$ for every $\alpha\in A_1$. Let $\mathfrak{U}_2=\{U_\alpha; \alpha\in A_2\}$ be a locally finite open covering of order $\leq n+1$ such that the diameter of each $U_\alpha(\alpha\in A_2)<1/2$ and $\mathfrak{U}_2$ refines $\mathfrak{V}_1$. Let $\mathfrak{F}_2=\{F_\alpha; \alpha\in A_2\}$ and $\mathfrak{V}_2=\{V_\alpha; \alpha\in A_2\}$ be respectively a closed covering and an open covering of R such that $U_\alpha\supset F_\alpha\supset V_\alpha$ for every $\alpha\in A_2$. Proceeding this procedure, we get a sequence of closed coverings $\mathfrak{F}_1, \mathfrak{F}_2,\cdots$ such that $\mathfrak{F}_1>\mathfrak{F}_2>\cdots$ and the diameter of each set of $\mathfrak{F}_i<1/i$ and the order of each $\mathfrak{F}_i\leq n+1$. For every i let us define a single-valued mapping $f_{i+1,i}$ of A_{i+1} with the discrete topology into A_i with the discrete one as follows: $f_{i+1,i}(\alpha)=\beta$ leads to $F_\alpha\subset F_\beta$. Let S_0 be the inverse limiting space obtained from $\{A_i; f_{i+1,i}\}$. Let R_0 be the subspace of S_0 such that $x=(\alpha_1,\alpha_2,\cdots)\in R_0$ if and only if

$\bigcap_{i=1}^{\infty}\{F_{\alpha_i}; (\alpha_1, \alpha_2, \cdots)\in S_0\}\neq\phi$. When $R\neq\phi$, we can see $R_0\neq\phi$. Let $f: R_0\to R$ be a transformation defined by $f(x)=\bigcap_{i=1}^{\infty}F_{\pi_i(x)}$. Then we can verify that f is a closed mapping of R_0 onto R such that $f^{-1}(y)$ consists of at most $n+1$ points.

Theorem 5 (Morita [2, Theorem 5.3] and Katětov [1, Theorem 3.4]). *Let R be a metric space. Then $\dim R\leq n(<\infty)$ if and only if R is the sum of $n+1$ subspaces R_i with $\dim R_i\leq 0$.*

Theorem 6 (Morita [2, Theorem 8.6] and Katětov [1, Theorem 3.4]). *Let R be a metric space. Then $\dim R=\operatorname{Ind} R$, where $\operatorname{Ind} R$ is the inductive dimension of R defined by means of the separation of closed sets.*

Theorem 7. *Let R be a metric space with $\dim R=n(<\infty)$. Then for every $\varepsilon>0$, there exists a locally finite closed covering $\mathfrak{F}$ of R of order $n+1$ such that the diameter of each set of $\mathfrak{F}<\varepsilon$ and that for any i, $1\leq i\leq n+1$, there exists a point of R at which the order of $\mathfrak{F}$ is i.*

Theorem 8. *Let R be a metric space with $\dim R\leq n(<\infty)$. Then there exist a dense subset A_0 of $\lim A_i=\lim\{A_i, f_{i+1,i}\}$, where A_i is the discrete space of indices, and a sequence of locally finite closed coverings $\mathfrak{F}_i=\{F_\alpha; \alpha\in A_i\}$, $i=1, 2,\cdots$, which satisfy the following conditions.*

(1) *The diameter of each set of $\mathfrak{F}_i<1/i$.*

(2) *The order of every $\mathfrak{F}_i\leq n+1$.*

(3) *For any i and any $\alpha\in A_i$,*
$$F_\alpha=\bigcup\{F_\beta; \beta\in A_{i+1}, f_{i+1,i}(\beta)=\alpha\}.$$

(4) *For any i and any s, $\dim\bigcap_{j=1}^{s}\{F_{\alpha(j)}, \alpha(1),\cdots,\alpha(s)$ are mutually different indices of $A_i\}\leq n-s+1$.*

Moreover if $\{\mathfrak{F}_i; i=1, 2,\cdots\}$ satisfies conditions (1), (2), (3), *then it satisfies condition* (4).

The first part of this theorem is implicitly stated in Morita [3].

Theorem 9. *Let R be a metric space and let $C_1, C_2,\cdots$ be countable closed sets of R with $\dim C_i<\infty$. Then there exist a dense subset A_0 of $\lim A_i=\lim\{A_i; f_{i+1,i}\}$, where A_i is the discrete space of indices, and a sequence of locally finite closed coverings $\mathfrak{F}_i=\{F_\alpha; \alpha\in A_i\}$, $i=1, 2,\cdots$, which satisfy the following conditions.*

(1) *The diameter of each set of $\mathfrak{F}_i<1/i$.*

(2) *For any i and any j, the order of $\mathfrak{F}_i\cap C_j\leq\dim C_j+1$.*

(3) *For any i and any $\alpha\in A_i$,*
$$F_\alpha=\bigcup\{F_\beta; \beta\in A_{i+1}, f_{i+1,i}(\beta)=\alpha\}.$$

(4) *For any i, s and t,*

$\dim\bigcap_{j=1}^{s}\{F_{\alpha(j)}\cap C_i; \alpha(1),\cdots,\alpha(s)$ *are mutually different indices of* $A_t\}\leq\dim C_i-s+1$.

Moreover if $\{\mathfrak{F}_i; i=1, 2, \cdots\}$ *satisfies conditions* (1), (2), (3), *then it satisfies condition* (4).

The first part of this theorem has been proved by Morita, though unpublished.

An analogue to Theorem 2 is also true.

Theorem 10. *Let R and S be metric spaces with* $\dim R \leq 0$ *and f an open mapping of R onto S such that* $f^{-1}(y)$ *is a finite set at every point* $y \in S$. *Then for any m, we have* $\dim\{y; |f^{-1}(y)|=m\} \leq 0$.

Using this theorem we get

Theorem 11. *Let R and S be metric spaces with* $\dim R \leq 0$. *If there exists an open mapping of R onto S such that* $f^{-1}(y)$ *is a finite set at every point* $y \in S$, *then* $\dim S \leq 0$.

References

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