115. On Certain Examples of the Crossed Product of Finite Factors. II

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In the preceding paper [1], we constructed an automorphism group of a hyperfinite continuous factor which is isomorphic to a given enumerably infinite group G and showed the automorphism group is outer if G is torsion free whereas we have remained open that [1, Theorem 2] is valid without exception. In this note we shall answer the question in affirmative, that is, we shall show the following

THEOREM. *There is an outer automorphism group of the hyperfinite continuous factor which is isomorphic to the given enumerable group.*

We use same notations as in [1]. X is the product space of $E_g=\{0, 1\}$ $(g\in G)$, Γ is the subset of X composed of the elements $x=[x_g]$ such that $x_g=0$ except for a finite number of g's. A measure m is defined naturally on X and a group of measure preserving transformations $\{T_\gamma \mid \gamma\in\Gamma\}$ is constructed isomorphically to Γ. Hence, by the Murray-von Neumann method, a hyperfinite continuous factor $\boldsymbol{A}$ is generated from operators $L_{\varphi(x)}$ ($\varphi(x)$ denotes bounded measurable functions on X) and U_γ $(\gamma\in\Gamma)$ on the Hilbert space $H=L^2(\Gamma\times X)$. Furthermore every element $g_0\in G$ gives a measure preserving transformation T_{g_0} on X such that

$$xT_{g_0}=[x_{g_0g}] \qquad \text{for } x=[x_g]$$

and so g_0 induces an automorphism of $\boldsymbol{A}$ such that

$$U_\gamma^{g_0}=U_{\gamma T_{g_0}}, \quad L_{\varphi(x)}^{g_0}=L_{\varphi(xT_{g_0})}.$$

These automorphisms give the automorphism group in question.

1. A lemma of I. M. Singer. I. M. Singer [2] has analized in detail inner automorphisms of finite factors constructed by the Murray-von Neumann method. Especially he studied the automorphisms which preserve the commutative subalgebra $\boldsymbol{L}$ generated from $\{L_{\varphi(x)}\}$. Favourably our automorphisms preserve $\boldsymbol{L}$ and his results prepare for us a way to the proof of the theorem.

LEMMA 1 (I. M. Singer [2, Lemma 2.2]). *If* (α) *the ergodic group of the measure preserving transformations* $\{T_\gamma \mid \gamma\in\Gamma\}$ *satisfies the condition:*

(∗) *for a measurable set E with positive measure and every T_γ there exists a subset F of E such that*

$$m(FT_\gamma \triangle F) \neq 0$$

where $\triangle$ denotes the symmetric difference and (β) *a unitary operator* $V \approx [[\xi_\gamma(x)]]$ *in* $\boldsymbol{A}$ (cf. [1, Theorom 2]) *induces an inner automorphism of* $\boldsymbol{A}$ *which preserves* $\boldsymbol{L}$, *then* $\xi_\gamma(x)$'*s satisfy the following conditions, where* $E_\gamma = \{x \mid \xi_\gamma(x) \neq 0\}$,

(i) $m(\bigcup_{\gamma \in \Gamma} E_\gamma) = 1$,

(ii) $m(E_\gamma \cap E_\delta) = 0$ *if* $\gamma \neq \delta$ $(\gamma, \delta \in \Gamma)$,

(iii) $m(E_\gamma T_\gamma \cap E_\delta T_\delta) = 0$ *if* $\gamma \neq \delta$,

(iv) $\sum_{\gamma \in \Gamma} |\xi_\gamma(x)| = 1$ *a.e. on* X, $|\xi_\gamma(x)| = 1$ *a.e. on* E_γ.

2. Proof of the theorem

LEMMA 2. *The measure preserving transformation T_γ on X satisfies the condition* (∗) *always.*

PROOF. Let E be a measurable set with positive measure. If $m(ET_\gamma \triangle E) \neq 0$, we may take $F = E$. Next we assume $m(ET_\gamma \triangle E) = 0$ and $\gamma = [\gamma_g]$,

$$\gamma_g = \begin{cases} 1 & \text{for } g = g_1, g_2, \cdots, g_n, \\ 0 & \text{otherwise.} \end{cases}$$

Let E_y be the set of elements $x = [x_g] \in E$ such that

$$x_g = y(g) \qquad \text{for} \quad g = g_1, g_2, \cdots, g_n$$

where $y(g)$ is a function defined on $\{g_1, g_2, \cdots, g_n\}$ taking values in $\{0, 1\}$. Then E decomposes into a finite number of mutually disjoint sets $E_{y_1}, E_{y_2}, \cdots, E_{y_m}$. At least one of these sets has positive measure and it is transformed onto another one by T_γ. Hence an E_{y_i} gives a desired set F. This proves the lemma.

Now we assume any automorphism $g_0 \in G$, which is not identity, is inner and a unitary operator $V \approx [[\xi_c(x)]]$ induces the automorphism. Then since

$$L^{g_0}_{\varphi(x)} = L_{\varphi(xT_{g_0})} \in \boldsymbol{L},$$

the automorphism preserves $\boldsymbol{L}$ and so we can utilize Singer's lemma for V.

We take up $\gamma^h = [\gamma^h_g]$ in Γ, where

$$\gamma^h_g = \begin{cases} 1 & \text{if } g = h, \\ 0 & \text{if } g \neq h. \end{cases}$$

As shown in [1, Theorem 2],

$$U_{\gamma^h} \approx [[\chi^{\gamma^h}_c(x)]], \qquad U^{g_0}_{\gamma^h} \approx [[\chi^{\gamma^h T_{g_0}}_c(x)]]$$

where $\chi^{\gamma^h}_c(x) \equiv \begin{cases} 1 & \text{if } c = \gamma^h \\ 0 & \text{otherwise,} \end{cases}$ $\quad \chi^{\gamma^h T_{g_0}}_c(x) \equiv \begin{cases} 1 & \text{if } c = \gamma^h T_{g_0} \\ 0 & \text{otherwise} \end{cases}$

and

$$U^{g_0}_{\gamma^h} V \approx [[\textstyle\sum_\gamma \chi^{\gamma^h T_{g_0}}_\gamma(x) \xi_{c+\gamma}(x+\gamma)]] = [[\xi_{c+\gamma^h T_{g_0}}(x + \gamma^h T_{g_0})]]$$

$$V U_{\gamma^h} \approx [[\textstyle\sum_\gamma \xi_\gamma(x) \chi_{c+\gamma}(x+\gamma)]] = [[\xi_{c+\gamma^h}(x)]].$$

Hence $U^{g_0}_{\gamma^h} V = V U_{\gamma^h}$ implies

$$\xi_{c+\gamma^h}(x) = \xi_{c+\gamma^h T_{g_0}}(x+\gamma^h T_{g_0}) \qquad \text{a.e.}$$

By Singer's result

$$|\xi_{c+\gamma^h}(x)| = \begin{cases} 1 & \text{a.e. on } E_{c+\gamma^h} \\ 0 & \text{a.e. on } X - E_{c+\gamma^h}. \end{cases}$$

Since m is Γ-invariant

$$m(E_{c+\gamma^h}) = \int_X |\xi_{c+\gamma^h}(x)|\, dx = \int_X |\xi_{c+\gamma^h T_{g_0}}(x+\gamma^h T_{g_0})|\, dx$$

$$= \int_X |\xi_{c+\gamma^h T_{g_0}}(x)|\, dx = m(E_{c+\gamma^h T_{g_0}}).$$

For an arbitrary element d_0 in Γ and different elements $h, h', h', \cdots$ in G, we choice elements $c, c', c'', \cdots$ in Γ such that

$$d_0 = c + \gamma^h = c' + \gamma^{h'} = c'' + \gamma^{h''} = \cdots,$$

then by the above result

$$m(E_{d_0}) = m(E_{c+\gamma^h T_{g_0}}) = m(E_{c'+\gamma^{h'} T_{g_0}}) = m(E_{c+\gamma^{h''} T_{g_0}}) = \cdots \quad .$$

On the other hand, since $c+\gamma^h = c' + \gamma^{h'}$,

$$c + \gamma^h T_{g_0} = c' + \gamma^{h'} T_{g_0} \quad \text{if and only if} \quad \gamma^h + \gamma^h T_{g_0} = \gamma^{h'} + \gamma^{h'} T_{g_0}.$$

$\gamma^h + \gamma^h T_{g_0}$ is an element $\gamma = [\gamma_g]$ in Γ such that

$$\gamma_g = \begin{cases} 1 & \text{if } g = h \text{ or } g_0^{-1} h \\ 0 & \text{otherwise.} \end{cases}$$

Similarly $\gamma^{h'} + \gamma^{h'} T_{g_0}$ is $\gamma' = [\gamma'_g]$ such that

$$\gamma'_g = \begin{cases} 1 & \text{if } g = h' \text{ or } g_0^{-1} h' \\ 0 & \text{otherwise.} \end{cases}$$

Hence

$\gamma^h + \gamma^h T_{g_0} = \gamma^{h'} + \gamma^{h'} T_{g_0}$ if and only if $h = h'$ or $h = g_0^{-1} h'$ and $h' = g_0^{-1} h$. Thus, neglecting null sets, $E_{c+\gamma^h T_{g_0}}$ is identical with at most one of $E_{c+\gamma^{h'} T_{g_0}}$, $E_{c+\gamma^{h''} T_{g_0}}, \cdots$ and disjoint with others. At any rate, an infinite number of sets among $E_{c+\gamma^h T_{g_0}}$, $E_{c+\gamma^{h'} T_{g_0}}$, $E_{c+\gamma^{h''} T_{g_0}}, \cdots$ are mutually disjoint with each other ignoring null sets. Since $m(X) = 1$, we can conclude $m(E_{d_0}) = 0$. This means $V = 0$, that is, the automorphism g_0 is not inner.

References

[1] M. Nakamura and Z. Takeda: On certain examples of the crossed product of finite factors. I, Proc. Japan Acad., **34**, 495–499 (1958).

[2] I. M. Singer: Automorphisms of finite factors, Amer. Jour. Math., **17**, 117–133 (1955).