114. *On Certain Examples of the Crossed Product of Finite Factors. I*

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The purpose of the present note is two folds: The first is to establish that the hyperfinite continuous factor has many groups of outer automorphisms and the second is to show that the crossed product of the hyperfinite continuous factor is not always hyperfinite, which may give a right to the abstract treatment of the crossed products of von Neumann algebras. As stated in Turumaru [5] or in the previous [2], the method of the factor construction due to Murray-von Neumann [1] is nothing but the crossed product of an abelian algebra. It may be expected that the crossed product construction on non-abelian algebras is useful in the future study of von Neumann algebras. The present note may be observed as the first step towards it.

We utilize Murray-von Neumann's notation calculation as possible to save the space for description. The terminology is the same as in the previous [2]. Thus, for example, every automorphism in this paper means always a $*$-automorphism of a von Neumann algebra acting on a separable Hilbert space.

1. **Hyperfinite continuous factor and its automorphism group.** The pair of numbers $\{0, 1\}$ becomes an additive group by mod. 2. Its Haar measure μ is such that $\mu(\{0\})=\frac{1}{2}$, $\mu(\{1\})=\frac{1}{2}$. For every element g of an enumerably infinite group G, we associate the group $E_g=\{0,1\}$ and its Haar measure μ_g. By X denotes the product space of $E_g(g\in G)$ and the integration on X means always the one by the product measure of μ_g. That is, X is the set of all systems $x=[x_g;\ g\in G]$ where each $x_g=0$ or 1 and the total measure of X is one. Next put Γ the set of those $x=[x_g;\ g\in G]$ for which $x_g=0$ except for a finite number of g's. By the group operation of each component, Γ becomes a group, i.e.

$$\gamma+\gamma'=[\gamma_g+\gamma'_g] \quad (\text{mod. } 2) \quad \text{for } \gamma=[\gamma_g],\ \gamma'=[\gamma'_g].$$

Furthermore, defining

$$xT_\gamma=x+\gamma=[x_g+\gamma_g] \quad \text{for} \quad x=[x_g]\in X,$$

we get a measure preserving, ergodic transformation group $\{T_\gamma \mid \gamma\in\Gamma\}$ on X [3, Lemma 7.5.1]. As easily seen, this $\{T_\gamma\}$ is isomorphic to Γ

as an abstract group. Since Γ is the set theoretical sum of countable family of finite subgroups, the von Neumann algebra constructed by the Murray-von Neumann method for the measure space X and the transformation group $\{T_\gamma \mid \gamma \in \Gamma\}$ is a hyperfinite continuous factor. This is summarized as follows:

Let H be the Hilbert space of complex valued function $F(\gamma, x)$ on $\Gamma \times X$ such that

$$\sum_{\gamma \in \Gamma} \int_X |F(\gamma, x)|^2 dx < \infty.$$

Defining

$$U_{\gamma_0} \cdot F(\gamma, x) = F(\gamma + \gamma_0, xT_{\gamma_0}) \quad \textit{for} \quad \gamma_0 \in \Gamma$$
$$L_\varphi \cdot F(\gamma, x) = \varphi(x) F(\gamma, x)$$

for bounded measurable function $\varphi(x)$ on X, we get bounded operators U_{γ_0}, L_φ on H and these operators $\{U_{\gamma_0} \mid \gamma_0 \in \Gamma\}$, $\{L_\varphi \mid \varphi \in L^\infty(X)\}$ generate a hyperfinite continuous factor $\boldsymbol{A}$ standardly acting on H.

For $x = [x_g] \in X$, $\gamma = [\gamma_g] \in \Gamma$, put $xT_{g_0} = [x_{g_0 g}]$ and $\gamma T_{g_0} = [\gamma_{g_0 g}]$ $(g_0 \in G)$, then G is faithfully represented as a measure preserving transformation group on X and as a permutation group of Γ respectively. Thus defining

$$U_{g_0} \cdot F(\gamma, x) = F(\gamma T_{g_0}, xT_{g_0}) \quad \text{for} \quad F(\gamma, x) \in H,$$

we get a unitary group on H, which is anti-isomorphic to G. Furthermore, by the definitions of T_{g_0}, T_{γ_0}

$$xT_{g_0}^{-1} T_{\gamma_0} T_g = xT_{(\gamma_0 T_{g_0})}.$$

Hence we get

$$U_{g_0} U_{\gamma_0} U_{g_0}^* \cdot F(\gamma, x) = U_{(\gamma_0 T_{g_0})} F(\gamma, x)$$
$$U_{g_0} L_\varphi U_{g_0}^* \cdot F(\gamma, x) = \varphi(xT_{g_0}) F(\gamma, x).$$

These show $U_{\gamma_0}^{g_0} \equiv U_{g_0} U_{\gamma_0} U_{g_0}^* \in \boldsymbol{A}$ and $L_\varphi^{g_0} \equiv U_{g_0} L_\varphi U_{g_0}^* \in \boldsymbol{A}$ and so every $g_0 \in G$ determines an automorphism of $\boldsymbol{A}$. Thus G is represented as an automorphism group of $\boldsymbol{A}$. This representation is faithful. Thus we obtain

THEOREM 1. *If G is an enumerably infinite abstract group, then there exists an automorphism group on a hyperfinite continuous factor which is isomorphic to G.*

2. Outer automorphism groups. As seen in the previous paper, outer automorphism group is important in the crossed product. We shall give a criterion for a group of automorphisms being outer.

THEOREM 2. *If G is a torsion free enumerably infinite group, that is, every element of G, except the unit, has infinte order, then the automorphism group of a hyperfinite continuous factor in Theorem 1 is outer.*

Proof. By Murray-von Neumann [1, Lemma 12.4.1], every element A in $\boldsymbol{A}$ is expressed as

$$A \approx [[\chi_c(x)]]_{x \in X,\ c \in \Gamma},$$

where $\chi_c(x)\in L^\infty(X)$, and we can study the algebraic operation in A with this expression.

For $f(x)\in L^2(X)$ and

$$\varphi_s(\gamma)=\begin{cases}1 & \text{if } \gamma=s\\ 0 & \text{if } \gamma\neq s,\end{cases}$$

$$U_{\gamma_0}\{f(x)\varphi_s(\gamma)\}=f(x+\gamma_0)\varphi_s(\gamma+\gamma_0)=\begin{cases}f(x+\gamma_0) & \text{if } \gamma+\gamma_0=s\\ 0 & \text{if } \gamma+\gamma_0\neq s.\end{cases}$$

Hence

$$U_{\gamma_0}\approx[[\chi_c^{\gamma_0}(x)]] \quad \text{where} \quad \chi_c^{\gamma_0}(x)=\begin{cases}1 & \text{if } c=\gamma_0\\ 0 & \text{if } c\neq\gamma_0.\end{cases}$$

Similarly

$$U_{\gamma_0}^{g_0}=U_{(\gamma_0T_{g_0})}\approx[[\chi_c^{\gamma_0T_{g_0}}(x)]].$$

If there exists a $V\in \boldsymbol{A}$ such that $X^{g_0}=VXV^*$ for every $X\in\boldsymbol{A}$, $V\approx[[\xi_c(x)]]$, we take $\gamma_0=[\gamma_g^0]$ where $\gamma_g^0=1$ if $g=1$ (the unit of G), $\gamma_g^0=0$ otherwise, then $U_{\gamma_0}^{g_0}V=VU_{\gamma_0}$ and

$$U_{\gamma_0}^{g_0}V\approx[[\Sigma_\gamma\chi_\gamma^{\gamma_0T_{g_0}}(x)\,\xi_{c+\gamma}(x+\gamma)]]=[[\xi_{c+\gamma_0T_{g_0}}(x+\gamma_0T_{g_0})]]$$

$$VU_{\gamma_0}\approx[[\Sigma_\gamma\xi_\gamma(x)\chi_{c+\gamma}^{\gamma_0}(x+\gamma)]]=[[\xi_{c+\gamma_0}(x)]].$$

Therefore,

$$\xi_{c+\gamma_0}(x)=\xi_{c+\gamma_0T_{g_0}}(x+\gamma_0T_{g_0}) \qquad \text{a.e.}$$

Since $U_{\gamma_0}^{g_0g_0}V=VU_{\gamma_0}^{g_0}$, by the similar calculation, we get

$$\xi_{c+\gamma_0T_{g_0}}(x)=\xi_{c+\gamma_0T_{g_0}T_{g_0}}(x+\gamma_0T_{g_0}T_{g_0}) \qquad \text{a.e.}$$

Hence

$$\xi_{c+\gamma_0}(x)=\xi_{c+\gamma_0T_{g_0}}(x+\gamma_0T_{g_0})=\xi_{c+\gamma_0T_{g_0}T_{g_0}}(x+\gamma_0T_{g_0}+\gamma_0T_{g_0}T_{g_0})=\cdots \qquad \text{a.e.}$$

By the assumption, G is torsion free and so if $g_0\neq1$, each element of $\{\gamma_0,\ \gamma_0T_{g_0},\ \gamma_0T_{g_0}T_{g_0},\ \cdots\}$ is different from others. For an arbitrary $d_0\in\Gamma$, there exists $c\in\Gamma$ such that $d_0=c+\gamma_0$. By the above equality

$$\int_X|\xi_{d_0}(x)|^2\,dx=\int_X|\xi_{c+\gamma_0}(x+\gamma_0)|^2\,dx=\int_X|\xi_{c+\gamma_0T_{g_0}}(x+\gamma_0T_{g_0})|^2\,dx$$

$$=\int_X|\xi_{c+\gamma_0T_{g_0}T_{g_0}})x+\gamma_0T_{g_0}+\gamma_0T_{g_0}T_{g_0})|^2\,dx=\cdots \quad .$$

On the other hand, the trace of V^*V is given by

$$\tau(V^*V)=\sum_{d\in\Gamma}\int_X|\xi_d(x)|^2\,dx<\infty.$$

This means

$$\int_X|\xi_{d_0}(x)|^2\,dx=0$$

i.e. $\xi_{d_0}(x)=0$ a.e. Thus the automorphism due to $g_0\in G$ is outer.

3. Crossed product. By Theorem 2, the crossed product $G\otimes A$ by the automorphism group G is a continuous finite factor if G is torsion free [2]. But the direct computation, we can show the crossed product $G\otimes\boldsymbol{A}$ is a continuous finite factor always.

Let $\boldsymbol{H}$ be the Hilbert space of the complex valued functions

$F(g, \gamma, x)$ on $G \times \Gamma \times X$ such that

$$\| F \|^2 = \sum_G \sum_\Gamma \int_X | F(g, \gamma, x) |^2 dx < \infty.$$

On $\boldsymbol{H}$, we define

$$\overline{L}_\varphi : \ \overline{L}_\varphi F(g, \gamma, x) = \varphi(x) F(g, \gamma, x) \quad \text{for} \quad \varphi(x) \in L^\infty(X),$$

$$\overline{U}_{\gamma_0}: \ \overline{U}_{\gamma_0} F(g, \gamma, x) = F(g, \gamma + \gamma_0, x T_{\gamma_0}) \quad \text{for} \quad \gamma_0 \in \Gamma,$$

$$\overline{U}_{g_0}: \ \overline{U}_{g_0} F(g, \gamma, x) = F(g g_0, \gamma T_{g_0}, x T_{g_0}) \quad \text{for} \quad g_0 \in G.$$

The von Neumann algebra generated by these operators is specially isomorphic to $G \otimes \boldsymbol{A}$ stated in the previous paper. The element

$$F(g, \gamma, x) = \begin{cases} 1 & \text{if } \gamma = 0 \text{ (unit of } \Gamma), \ g = 1 \text{ (unit of } G) \\ 0 & \text{otherwise} \end{cases}$$

gives a trace element of $G \otimes \boldsymbol{A}$ [2, 5].

Now in $\Delta = G \times \Gamma = \{(g, \gamma) \mid g \in G, \gamma \in \Gamma\}$, we define

$$(g, \gamma)(h, \delta) = (gh, \gamma T_h + \delta),$$

then Δ becomes a group by this product. The unit of Δ is $(1, 0)$. The inverse of $(g, \gamma) = (g^{-1}, \gamma T_{g^{-1}})$. (g, γ) decomposes into $(g, 0)(1, \gamma)$.

Since both G and Γ are identified with measure preserving transformation groups on X, by the correspondence $(g, \gamma) \to T_g T_\gamma$, Δ itself is mapped isomorphically to the measure preserving transformation group on X. That is, Δ is seen as the transformation group on X composed by the measure preserving transformations belonging to G and Γ. As Γ acts ergodic on X, Δ itself acts ergodic too. The von Neumann algebra $\boldsymbol{R}(X, \Delta)$ constructed by the Murray-von Neumann method for the measure space X and the measure preserving transformation group Δ is a continuous finite factor [4, pp. 143–144].

THEOREM 3. *$G \otimes \boldsymbol{A}$ is a continuous finite factor.*

Proof. By the construction, the Hilbert space $\mathfrak{H}$ on which $\boldsymbol{R}(X, \Delta)$ acts is given by functions $F(\delta, x)$ on $\Delta \times X$ such that

$$\sum_{\delta \in \Delta} \int_X | F(\delta, x) |^2 dx < \infty.$$

Since $\delta = (g, \gamma) \in \Delta = G \times \Gamma$,

$$\sum_\delta \int_X | F(\delta, x) |^2 dx = \sum_g \sum_\gamma \int_X | F((g, \gamma), x) |^2 dx < \infty.$$

$\boldsymbol{R}(X, \Delta)$ is generated from operators such that

$$\mathfrak{L}_\varphi : \ \mathfrak{L}_\varphi \cdot F(\delta, x) = \varphi(x) F((g, \gamma), x) \qquad (\varphi \in L^\infty(X))$$

$$\mathfrak{U}_{\delta_0}: \ \mathfrak{U}_{\delta_0} \cdot F(\delta, x) = F(\delta \delta_0, x T_{\delta_0}) = F((g g_0, \gamma T_{g_0} + \gamma_0), x T_{g_0} T_{\gamma_0}) \quad (\delta_0 \in \Delta).$$

For $\delta_0 = (1, \gamma_0)$, we put $\mathfrak{U}_{\delta_0} = \mathfrak{U}_{\gamma_0}$, then

$$\mathfrak{U}_{\gamma_0} \cdot F(\delta, x) = F((g, \gamma + \gamma_0), x T_{\gamma_0}).$$

For $\delta_0 = (g_0, 0)$, we put $\mathfrak{U}_{\delta_0} = \mathfrak{U}_{g_0}$, then

$$\mathfrak{U}_{g_0} \cdot F(\delta, x) = F((g g_0, \gamma T_{g_0}), x T_{g_0}).$$

Therefore we define a unitary transformation V from $\mathfrak{H}$ onto $\boldsymbol{H}$ such that

$$V: \quad F(\delta, x) \equiv F((g, \gamma), x) \rightarrow F(g, \gamma, x),$$

then

$$V^{-1}\overline{L}_{\varphi}V=\mathfrak{L}_{\varphi}, \quad V^{-1}\overline{U}_{\gamma_0}V=\mathfrak{U}_{\gamma_0}, \quad V^{-1}\overline{U}_{g_0}V=\mathfrak{U}_{g_0}$$

and so $G\otimes \boldsymbol{A}$ is specially isomorphic to $\boldsymbol{R}(X, \Delta)$.

REMARK. It is probable that the automorphism group constructed in Theorem 1 is outer always but it is not certain for the present authors.

ADDED IN PROOF. After this note is presented, the question is solved affirmatively. This is shown in the next paper.

The construction of $\boldsymbol{R}(X, \Delta)$ is the same with Pukánszky's factor of type III. He proved that $\boldsymbol{R}(X, \Delta)$ has no *Property L* for the free group with two generators [4]. Thus

THEOREM 4. *The crossed product $G\otimes \boldsymbol{A}$ of the hyperfinite continuous factor $\boldsymbol{A}$ by the free group with two generators G is non-hyperfinite, continuous finite factor.*

REMARK. By the same construction for a finite group G, the resulting von Neumann algebra $G\otimes \boldsymbol{A}$ is not always a factor. In fact for $G=\{0, 1\}$, $G\otimes \boldsymbol{A}$ is exactly non-factor because the representing space of $G\otimes \boldsymbol{A}$ is of dimension 32 and so any finite factor can not act on it standardly. The reason of this fact is due to that Δ is not an m-group [1, p. 195] for this case.

References

[1] F. Murray and J. von Neumann: Rings of operators, Ann. Math., **37**, 116–229 (1936).

[2] M. Nakamura and Z. Takeda: On some elementary properties of the crossed products of von Neumann algebras, Proc. Japan Acad., **34**, 489–494 (1958).

[3] J. von Neumann: On infinite direct products, Compositio-Math., **6**, 1–77 (1938–1939).

[4] L. Pukánszky: Some examples of factors, Publ. Math., Debrecen, **4**, 135–156 (1956)

[5] T. Turumaru: On the crossed product of operator algebras (to appear in Tôhoku Math. Jour.).