113. *On Some Elementary Properties of the Crossed Products of von Neumann Algebras*

By Masahiro NAKAMURA
Osaka University of Liberal Arts and Education
and Zirô TAKEDA
Ibaragi University

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1. Many mathematical models would be employed by von Neumann in his investigation on the rings of operators. The theories of quantum mechanics, infinite groups, integration, measure preserving transformations and so on would be found among them. However, the classical theory of simple algebras would play an eminent rôle in his monumental "rings of operators" series, since the von Neumann algebras are recognized by himself as an infinite dimensional hypercomplex numbers in his earlier paper.

In this point of view, it is curious that the notion of crossed product is paid a little attension in the literatures, although it plays an essential rôle in the theory of simple algebras. As the authors concern, the first abstract definition of the crossed product of von Neumann algebras is introduced by Turumaru in 1955 by a seminar conversation, who also pointed out that the examples of factors due to Murray-von Neumann [2] is nothing but the crossed product of an abelian algebra by an ergodic automorphism group. However, the further development delayed, since no security existed that the crossed product of a factor produces an another new one.

In the succeeding paper [3], it will be proved that the crossed product of the hyperfinite continuous factor is not isomorphic to the original when the group of automorphisms is suitably restricted. Therefore, the purpose of the present note is to develope some elementary properties of the crossed product of a finite factor along the line that some well-known theorems of simple algebras are still valid for finite factors. Incidentally, in the end of the note, a theorem of Murray-von Neumann concerning the example construction will be given an alternative simpler proof basing on the same idea.

2. At the beginning, we may make a few remarks. The terminology of J. Dixmier [1] will be used without any explanation unless the contrary is stated (for example, von Neumann algebra and the hyperfiniteness will be used instead of W^*-algebras and the approximate finiteness respectively). Moreover, each von Neumann algebra of the present note will be assumed to act on a separable Hilbert space.

We shall begin with a few preliminaries about the automorphisms on certain von Neumann algebras. Since $*$-operation is considered as an algebraic operation on the algebras, automorphisms mean always $*$-automorphisms in this paper. An automorphism g of a von Neumann algebra A is *outer* if there exists no unitary operator $u\in A$ such that $x^g=uxu^*$ for $x\in A$ (x^g is the image of x due to the automorphism g). A group of automorphisms is outer if every automorphism except the unit is outer. Then the following lemma, which is clearly a direct analogue of a classical theorem on simple algebras, is fundamental in the present note:

LEMMA 1. *If A is a finite factor with an outer automorphism g, and if*

$$x^g a = ax, \tag{1}$$

for all $x\in A$, then a vanishes.

Proof. (1) implies that $I=aA$ is a two-sided ideal. By the well-known proposition of von Neumann [4, p. 25] a finite factor is simple, whence $I=A$ unless $a=0$, and so a is regular unless $a=0$. This implies, by a proposition of Dixmier [1, p. 15] that the automorphism g is inner, which is a contradiction unless $a=0$.

An abelian analogue of Lemma 1 is the following

LEMMA 2. *If A is an abelian von Neumann algebra with an automorphism g, then the set I of all a which satisfies* (1) *for any $x\in A$ is a closed ideal element-wise invariant under g. Therefore, if moreover g is ergodic in the sense that the invariant elements of A are only the scalars, then I vanishes.*

Proof. It is not difficult to see that I is a closed ideal which does not contain the scalars whenever g is non-trivial. By $(ab)^g=a^gb^g=a^gb=ab$ for $a,b\in I$, it is obvious that I^2 is element-wise invariant under g. By the well-known theorem of Segal I^2 is dense in I, whence I is element-wise invariant. The remainder of the lemma is now obvious.

In §5, Lemma 2 will be used as an alternative of [2, Lemma 12.2.3].

3. Following after Turumaru [6], we shall introduce the crossed product $G\otimes A$ of a von Neumann algebra A with a faithful normal trace τ by an enumerable group G of outer automorphisms.

Conveniently we denote a function on G with the value a_g in A at g by $\Sigma_g\, g\otimes a_g$. Let D be the set of all functions $\Sigma_g\, g\otimes a_g$ such that $a_g=0$ except a finite number of g's. Besides the usual addition and scalar multiplication the following operations will be introduced in the set D:

$$(\Sigma_g\, g\otimes a_g)(\Sigma_h\, h\otimes b_h)=\Sigma_{g,h}\, gh\otimes a_g^h b_h, \tag{2}$$

$$(\Sigma_g\, g\otimes a_g)^*=\Sigma_g\, g^{-1}\otimes a_g^{g^{-1}*}. \tag{3}$$

It is not hard to show that D is a $*$-algebra by the above computation rules. For the convenience, A and G will be considered as the subsets of D identifying a and g with $1\otimes a$ and $g\otimes 1$ respectively (especially, $1\otimes 1$ is considered as the identity of A and G). In D, a trace τ which is the extension of the given trace of A will be introduced by

$$(4)\qquad \tau(g\otimes a_g)=\begin{cases}\tau(a_g) & \text{for } g=1,\\ 0 & \text{otherwise,}\end{cases}$$

$$(5)\qquad \tau(\Sigma_g g\otimes a_g)=\Sigma_g \tau(g\otimes a_g).$$

Turumaru proved that τ is faithful on D, and that the Gelfand-Neumark representation space by τ is nothing but $G\otimes H$ in the sense of Umegaki [7] where H is the standard representation space of A by its trace, and consequently that D is faithfully represented on $G\otimes H$. Turumaru defined that the *crossed product* $G\otimes A$ *of* A *by* G (*in* τ) is the weak closure of D on $G\otimes H$ (naturally, the C^*-*crossed product* is also defined by the *uniform closure* of D which is not the subject in the present note). Since τ is a faithful trace on D, $G\otimes A$ is of finite type and embedded in $G\otimes H$.

Since an element of $G\otimes A$ can be considered as an element of $G\otimes H$, it has an expression $\Sigma_g g\otimes k_g (k_g\in H)$ which will be called the *Fourier expansion* and each k_g will be called the *g-th Fourier coefficient* or simply *g-coefficient*. Clearly, k_g is uniquely determined by the given element. For a pair of elements of $G\otimes H$ whose all coefficients a_g and b_g belong to A embedded in H, the inner product will be described by the following

$$(6)\qquad \langle \Sigma_g g\otimes a_g \mid \Sigma_h h\otimes b_h\rangle=\Sigma_g \tau(a_g b_g^*),$$

especially the Hilbert space norm is

$$(7)\qquad \|\Sigma_g g\otimes a_g\|_2^2=\Sigma_g \tau(a_g a_g^*).$$

These will be considered as a generalization of the results of [2, § 12.4]. In the next section, it is shown that all g-coefficients of an element belonging to $G\otimes A$ are in A.

4. For the convenient use in the below, we shall here introduce and make a few remarks about the (conditional) *expectation* due to Umegaki [8], which is an operation defined on a finite von Neumann algebra A with a faithful normal trace τ. If B is a (weakly closed self-adjoint) subalgebra of A, then *the expectation conditioned by* B is a mapping $x\to x^+$ from A into B satisfying

$$(8)\qquad \tau(x^+y)=\tau(xy)\qquad \text{for all } y\in B.$$

Umegaki observed that the expectation is a positive linear normal transformation of A onto B which satisfies moreover

$$(9)\qquad (x^+y)^+=x^+y^+=(xy^+)^+,$$

or equivalently, the B-module homomorphy;

$$(10)\qquad (xb)^+=x^+b,\quad (bx)^+=bx^+\qquad \text{for any } b\in B.$$

Umegaki characterized it conversely that a positive linear normal idempotent B-module endomorphism of A is the (conditional) expectation conditioned by B, which is recently sharpened by Tomiyama [5] that the expectation is a projection of norm one having the range B and satisfying (10).

An another nature of the conditional expectation is a projection of the standard representation space H (by τ) whose range is the metric hull of B. This statement follows from the following two relations:

$$\| x^{\div} \|_2^2 = | \tau(x^{\div} x^{\div *}) | = | \tau(x x^{\div *}) | \leqq \tau(x x^*)^{\frac{1}{2}} \tau(x^{\div} x^{\div *})^{\frac{1}{2}},$$
$$\langle x^{\div} | y \rangle = \tau(x^{\div} y^*) = \tau(x^{\div} y^{*\div}) = \tau(x y^{\div *}) = \langle x | y^{\div} \rangle.$$

Since $G \otimes A$ has a faithful normal trace by the above, each (von Neumann) subalgebra B allows the expectation conditioned by B, especially there exists the expectation ε conditioned by A itself. Since a_1 is the expectation of $\Sigma_g g \otimes a_g$ of D conditioned by A, that is, $(\Sigma_g g \otimes a_g)^\varepsilon = a_1$, the strong or weak continuity (in the unit sphere) of the expectation (cf. [8]) implies that the above equality is valid for all elements of $G \otimes A$, whence the 1-coefficient of an element of $G \otimes A$ belongs to A. From this it follows that the g-coefficient of an element $\Sigma_g g \otimes a_g$ of $G \otimes A$ belongs to A since $g^{-1}(\Sigma_h h \otimes a_h)$ belongs to $G \otimes A$.

LEMMA 3. *If B is a finite factor with a subfactor A, G is an enumerable group of unitary operators of B each of which induces an outer automorphism $x^g = g^* x g$ of A unless $g=1$ and furthermore A and G generate B, then $B = G \otimes A$.*

Proof. By the assumption, $x^g g^\varepsilon = g^\varepsilon x$ where ε is the expectation conditioned by A for any element x of A, whence Lemma 1 implies $g^\varepsilon = 0$, and so B and G are orthogonal in the sense that

$$\langle g | b \rangle = \tau(g b^*) = 0$$

unless $g=1$. Hence (4) is satisfied by $g a_g$ instead of $g \otimes a_g$. Therefore the correspondence $g a_g \leftrightarrow g \otimes a_g$ gives an isometry between the metric hull of B and $G \otimes H$. Since the computation rules (2) and (3) are satisfied by $\Sigma_g g a_g$ instead of $\Sigma_g g \otimes a_g$, these imply the spatial isomorphism between B and $G \otimes A$.

We shall finally notice that Lemma 3 shows the regularity of A (in the sense of Dixmier) in its crossed product $G \otimes A$.

5. Now, we are able to trace the analogy between the crossed products of finite factors and simple algebras (of finite rank). First at all, we shall prove the following

THEOREM 1. *The crossed product of a finite factor by an enumerable group of outer automorphisms is also a finite factor.*

Proof. Obviously, we need only to show that $G \otimes A$ is a factor. If e is a non-trivial central projection of $G \otimes A$, with the g-coefficient

a_g, then the g-coefficient of xe and ex coincides for any $x \in A$, that is, $x^g a_g = a_g x$ for all $x \in A$. By Lemma 1, hence $a_g = 0$ unless $g = 1$, which is clearly impossible.

REMARK 1. The theorem seems to state for all finite factors, however, I_n factors are automatically excluded since they have no outer automorphisms by a theorem of Kaplansky. Compare with Remark of Theorem 4 of [3].

REMARK 2. A proof of the theorem is recently obtained by N. Suzuki, who seems the first to prove the theorem. Although Suzuki and the authors proved it independently, it seems to the authors that their proofs are same in spirit.

THEOREM 2. *If A is a finite factor with an enumerable outer automorphism group G, then each von Neumann subalgebra B of $G \otimes A$ containing A is the crossed product $F \otimes A$ of A by a subgroup F of G up to isomorphism.*

Proof. Let $h^{\div}$ be the expectation of $h \in G$ conditioned by B (where h is seen as an element of $G \otimes A$), and let a_g be the g-coefficient of $h^{\div}$. Then

$$xh^{\div} = (xh)^{\div} = (hx^h)^{\div} = h^{\div} x^h \qquad \text{for each } x \in A$$

implies $a_g = 0$ by Lemma 1 comparing their g-coefficients unless $g = h$. Hence either $h = h^{\div} \in B$ or $h^{\div} = 0$. Let F be the set of all $h \in B$. Then it is obvious that F is a subgroup of G. The remainder of the proof follows from Lemma 3 or the direct application of the definition.

THEOREM 3. *In the crossed product $G \otimes A$ of a finite factor A with an enumerable outer automorphism group G, the lattice of all subfactors containing A is isomorphic to the lattice of all subgroups of G.*

Proof. Since by Theorem 2 each subfactor B containing A is the crossed product of A by a subgroup F of G, and since each subgroup F produces a subfactor $F \otimes A$ by Theorem 1, subfactor containing A corresponds to subgroup of G. By Theorem 2, the correspondence is determined by the inclusion $F \leqq B$ and the generation, it is clearly one-to-one. This proves the theorem since the order is clearly preserved under the correspondence.

6. Incidentally, we shall describe here an abelian analogue of the preceding section, which gives an alternative (but different to Turumaru [6]) abstract proof of the following well-known

THEOREM 4 (Murray-von Neumann). *If A is an abelian von Neumann algebra with a faithful normal trace and G is an ergodic group of enumerable outer automorphisms preserving the trace, then $G \otimes A$ is a finite factor in which A is maximally abelian.*

Proof will be carried out similar to that of Theorem 1. If a non-zero z (in $G \otimes A$) commutes with A in element-wise, and if a_g is the g-coefficient of z, then the g-coefficients of both sides of $xz=zx$ (for any $x\in A$) satisfy the hypothesis of Lemma 2, whence $a_g=0$ unless $g=1$ since G is ergodic, that is, $z=1\otimes a_1\in A$, which proves the second half of the theorem.

Let $a=a_1$ for the convenience. If a is central in $G\otimes A$, then

$$ga=ag=ga^g \qquad \text{for any } g\in G,$$

whence $a=a^g$ or a is a scalar by the ergodicity of G, which proves the first half of the theorem.

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