110. On Determination of the Class of Saturation in the Theory of Approximation of Functions

By Gen-ichiro SUNOUCHI and Chinami WATARI

(Comm. by K. KUNUGI, M.J.A., Oct. 13, 1958)

1. Introduction. Let $f(x)$ be an integrable function, with period 2π and let its Fourier series be

$$(1)\qquad \frac{a_0}{2}+\sum_{k=1}^{\infty}(a_k\cos kx+b_k\sin kx)\equiv\sum_{k=0}^{\infty}A_k(x).$$

Let $g_k(n)$ $k=1,2,\cdots$ be the summating function and consider a family of transforms of (1) of a summability method G,

$$(2)\qquad P_n(x)=\frac{a_0}{2}+\sum_{k=1}^{\infty}g_k(n)(a_k\cos kx+b_k\sin kx)$$

where the parameter n needs not be discrete.

If there are a positive non-increasing function $\varphi(n)$ and a class K of functions in such a way that

(I) $\quad \|f(x)-P_n(x)\|=o(\varphi(n))$[1] implies $f(x)=$constant;

(II) $\quad \|f(x)-P_n(x)\|=O(\varphi(n))$ implies $f(x)\in K$;

(III) for every $f(x)\in K$, one has $\|f(x)-P_n(x)\|=O(\varphi(n))$,

then it is said that the method of summation G is saturated with order $\varphi(n)$ and its class of saturation is K. This definition is due to J. Favard [2].

The purpose of this article is to determine the order and the class of saturation for several familiar summation methods. M. Zamansky [5] has solved this problem for the method of Cesàro-Fejér, with respect to the space (C) of continuous functions; P. L. Butzer [1] studied the cases of methods of Abel-Poisson and Gauss-Weierstrass, employing the theory of semi-groups, but, as he made use of the regularity of the spaces (L^p) $p>1$, he left the question open for the spaces (C) and (L).

We give here a direct method to determine the class of saturation for general method of summability, with respect to the spaces (C) and (L^p) $p\geqq 1$. The above condition (I) is easily verified and the condition (III) is proved by so-called singular integral method. The inverse problem (II) is the key point of this paper.

2. The inverse problem. Let us write $\Delta_n(x)=f(x)-P_n(x)$ and suppose that there are positive constants c, r and ρ such that

$$(3)\qquad \lim_{n\to\infty}n^r(1-g_k(n))=ck^{\rho}\,[2]\qquad (k=1,2,\cdots).$$

1) The norm means (C)- or (L^p)-$(p\geqq 1)$ norm.

2) To fix the ideas, we take the limit as $n\to\infty$; but, as is easily seen, the following arguments remain valid, with appropriate modifications, in other cases (see Theorem 2 below).

(i) If $||\Delta_n(x)||=o(n^{-r})$, then we have

$$a_k(1-g_k(n))=\frac{1}{\pi}\int_{-\pi}^{\pi}\Delta_n(x)\cos kx\,dx=o(n^{-r}),\ (k=1,2,\cdots)$$

and, comparing this with (3), we see

$$a_k=0 \text{ and similarly } b_k=0 \quad (k=1,2,\cdots)$$

and consequently we have $f(x)=a_0/2$. Thus the condition (I) is verified.

(ii) Suppose now $||\Delta_n(x)||=O(n^{-r})$ and let $N<n$. Taking the N-th arithmetic mean $\sigma_N[x;\Delta_n]$ of the series

$$(4) \qquad \Delta_n(x)\sim\sum_{k=1}^{\infty}(1-g_k(n))A_k(x),$$

we have

$$\sigma_N[x;\Delta_n]=\sum_{k=1}^{N}(1-g_k(n))A_k(x)\Big(1-\frac{k}{N+1}\Big).$$

Because it is well known that $||\sigma_N[x;F]||\leqq||F||$ (for the spaces (C) and (L), this is trivial; for (L^p) $p>1$, we have only to apply Jensen's inequality), our hypothesis on $\Delta_n(x)$ yields

$$\Big\|\sum_{k=1}^{N}(1-g_k(n))A_k(x)\Big(1-\frac{k}{N+1}\Big)\Big\|=O\Big(\frac{1}{n^r}\Big)$$

in other words

$$\Big\|\sum_{k=1}^{N}n^r(1-g_k(n))A_k(x)\Big(1-\frac{k}{N+1}\Big)\Big\|=O(1),$$

from which it results that, evidently for the space (C) and by means of Fatou's lemma for (L^p) $p\geqq1$,

$$\Big\|\sum_{k=1}^{N}\lim_{n\to\infty}n^r(1-g_k(n))A_k(x)\Big(1-\frac{k}{N+1}\Big)\Big\|=O(1)$$

that is to say

$$\Big\|\sum_{k=1}^{N}k^{\rho}A_k(x)\Big(1-\frac{k}{N+1}\Big)\Big\|=O(1).$$

Denoting by $f^{[\rho]}(x)$ the trigonometric series $\sum_{k=1}^{\infty}k^{\rho}A_k(x)$, we see that this is nothing but $||\sigma_N[x;f^{[\rho]}]||=O(1)$, and the latter is equivalent respectively to

$f^{[\rho]}(x)$ is the Fourier series of a bounded function (for the space (C))

$f^{[\rho]}(x)$ is the Fourier series of a function in (L^p) (for the space (L^p), $p>1$)

$f^{[\rho]}(x)$ is the Fourier-Stieltjes series of a function of bounded variation (for the space (L)).

See for example [7, §§ 4, 31–4, 33].

3. The method of Cesàro-Fejér summation

In this case we have

$$P_n(x)=\sum_{k=0}^{n}\Big(1-\frac{k}{n+1}\Big)A_k(x)=\frac{1}{2(n+1)\pi}\int_{-\pi}^{\pi}f(x+t)\Big\{\frac{\sin(n+1)t/2}{\sin t/2}\Big\}^2dt$$

and

$$g_k(n)=\left(1-\frac{k}{n+1}\right), \quad \lim_{n\to\infty} n(1-g_k(n))=k.$$

The considerations of the preceding section yield

(i) if $\|\Delta_n(x)\|=o(1/n)$, we have $f(x)=\text{constant}$;

(ii) if $\|\Delta_n(x)\|=O(1/n)$, we have

$\tilde{f}'(x)\in B$ i.e. $\tilde{f}(x)\in \text{Lip}\, 1$ (for the space (C))

$\tilde{f}'(x)\in L^p$ i.e. $\tilde{f}(x)\in \text{Lip}\,(1, p)$ (for the space (L^p), $p>1$)

$\tilde{f}'(x)\in S$ i.e. $\tilde{f}(x)\in BV$ (for the space (L))

respectively. The inverse is known to be true, see A. Zygmund [6]. Thus we have

Theorem 1. *The method of Cesàro-Fejér summation is saturated; its order of saturation is n^{-1}, its class of saturation is the class of functions $f(x)$ for which*

$\tilde{f}(x)\in \text{Lip}\, 1$ *(for the space (C))*

$\tilde{f}'(x)\in L^p$ *or* $f'(x)\in L^p$ *(for the space (L^p), $\infty>p>1$)*

$\tilde{f}(x)\in BV$ *(for the space (L)),*

respectively.

In a manner similar to that in which we have proved the above theorem, we may show the following theorems.

The Abel-Poisson mean of $\mathfrak{S}[f]$ is

$$P_r(x)=\sum_{k=0}^{\infty} A_k(x)\, r^k=\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x+t)\frac{(1-r^2)}{1-2r\cos t+r^2}\, dt \quad (0\leqq r<1)$$

and $g_k(r)=r^k$. Thus we have

Theorem 2. *The method of Abel-Poisson summability is saturated; its order of saturation is $(1-r)$, its class of saturation is identical with that of the method of Cesàro-Fejér summability.*

The Riesz mean (R, n^ρ, λ) of $\mathfrak{S}[f]$ is

$$R_n(x)=\sum_{k=0}^{n}\left(1-\left(\frac{k}{n}\right)^\rho\right)^\lambda A_k(x) \quad \text{and} \quad g_k(n)=\left(1-\left(\frac{k}{n}\right)^\rho\right)^\lambda.$$

Theorem 3. *The method of Riesz summability (R, n^ρ, λ) is saturated; its order of saturation is $n^{-\rho}$, its class of saturation is the class of functions $f(x)$ for which*

$f^{[\rho]}(x)\in B$ *(for the space (C))*

$f^{[\rho]}(x)\in L^p$ *(for the space (L^p), $1<p<\infty$)*

$f^{[\rho]}(x)\in S$ *(for the space (L))*

where $f^{[\rho]}(x)$ denotes the trigonometric series $\sum_{k=1}^{\infty} k^\rho A_k(x)$.

Corollary. *If ρ is a positive integer, the class of saturation of the method of Riesz summability (R, n^ρ, λ) is the class of those functions $f(x)$ for which*

$f^{(\rho-1)}(x)\in \text{Lip}\, 1$ *if ρ is even* *(for the space (C))*

$\tilde{f}^{(\rho-1)}(x)\in \text{Lip}\, 1$ *if ρ is odd*

$f^{(\rho)}(x) \in L^p$ (*for the space* (L^p), $1<p<\infty$))

$f^{(\rho-1)}(x) \in BV$ *if* ρ *is even* (*for the space* (L)).

$\tilde{f}^{(\rho-1)}(x) \in BV$ *if* ρ *is odd*

The Gauss-Weierstrass integral of $f(x)$ is

$$W(x;\xi)=\sum_{n=0}^{\infty} e^{-k^2\xi/4}A_k(x)=\sqrt{\frac{\pi}{\xi}}\int_{-\pi}^{\pi} f(x+t)\, e^{-t^2/\xi}\, dt$$

and $g_k(\xi)=e^{-k^2\xi/4}$

Theorem 4. *The method of approximation by the Gauss-Weierstrass integral is saturated; its order of saturation is* ξ*; its class of saturation is the class of functions* $f(x)$ *for which*

$f'(x) \in \mathrm{Lip}\, 1$ (*for the space* (C))

$f''(x) \in L^p$ (*for the space* (L^p), $1<p<\infty$)

$f'(x) \in BV$ (*for the space* (L))

respectively.

Since the Bernstein-Rogosinski mean of $\mathfrak{S}[f]$ is defined by

$$B_n(x)=\frac{1}{2}\left\{S_n\left(x+\frac{\pi}{2n+1}\right)+S_n\left(x-\frac{\pi}{2n+1}\right)\right\}$$

$$=A_0+\sum_{k=1}^{n}\cos\frac{k\pi}{2n+1}A_k(x)$$

we have $g_k(n)=\cos\dfrac{k\pi}{2n+1}$ and

Theorem 5. *The method of approximation by the Bernstein-Rogosinski mean of* $\mathfrak{S}[f]$ *is saturated; its order of saturation is* n^{-2}*, and its class of saturation is the class of those functions* $f(x)$ *for which*

$f'(x) \in \mathrm{Lip}\, 1$ (*for the space* (C))

$f''(x) \in L^p$ (*for the space* (L^p), $1<p<\infty$)

$f'(x) \in BV$ (*for the space* (L))

respectively.

Since the integral of de la Vallée Poussin is defined by

$$V_n(x)=\frac{h_n}{2\pi}\int_{-\pi}^{\pi} f(x+t)\cos^{2n}\frac{t}{2}\,dt$$

$$=\sum_{k=0}^{n}\frac{(n!)^2}{(n-k)!\,(n+k)!}A_n(x) \qquad \left(h_n=\frac{2n(2n-2)\cdots 4\cdot 2}{(2n-1)(2n-3)\cdots 3\cdot 1}\right),$$

$$g_k(n)=\frac{(n!)^2}{(n-k)!\,(n+k)!}=1-\frac{k^2}{n}+O\left(\frac{1}{n^2}\right),$$

we have, as the answer to a problem proposed by P. L. Butzer [1],

Theorem 6. *The method of approximation by the integral of de la Vallée Poussin is saturated; its order of saturation is* n^{-1}*, its class of saturation is the class of functions* $f(x)$ *for which*

$f'(x) \in \mathrm{Lip}\ 1$ (*for the space* (C))

$f''(x) \in L^p$ (*for the space* (L^p), $1<p<\infty$)

$f(x) \in BV$ *(for the space* (L)*)*

respectively.

The integral of Jackson-de la Vallée Poussin is defined by

$$I_n(x) = \frac{\tau_4}{2\pi}\int_{-\infty}^{\infty} f\left(x+\frac{2t}{n}\right)\frac{\sin^4 t}{t^4}\,dt \qquad \left(\tau_4 = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\sin^4 t}{t^4}\,dt\right)$$

$$= \sum_{k=0}^{2n-1} h\left(\frac{k}{n}\right) A_k(x)$$

where

$$h(x) = \begin{cases} 1-\dfrac{3}{2}x^2+\dfrac{3}{4}|x|^3 & |x| \leqq 1 \\ \dfrac{1}{4}(2-|x|)^3 & 1 \leqq |x| \leqq 2 \\ 0 & |x| \geqq 2. \end{cases}$$

Theorem 7. *The method of approximation by the Jackson-de la Vallée Poussin integral is saturated; its order of saturation is* n^{-2}, *its class of saturation is the class of function* $f(x)$ *for which*

$f'(x) \in \mathrm{Lip}\ 1$ *(for the space* (C)*)*

$f''(x) \in L^p$ *(for the space* (L^p), $1 < p < \infty$*)*

$f(x) \in BV$ *(for the space* (L)*)*

respectively.

The detailed proof of these theorems will appear in another periodical.

The problem (III) of these singular integrals are well known (see B. Sz. Nagy [3], I. P. Natanson [4]).

References

[1] P. L. Butzer: Math. Ann., **133**, 410–425 (1957).
[2] J. Favard: Colloques d'Analyse Harmonique Nancy, Paris, **15**, 97–110 (1949).
[3] B. Sz. Nagy: Hungarica Acta Math., **1**, 1–39 (1948).
[4] I. P. Natanson: Konstruktive Funkionentheorie, Berlin (1955).
[5] M. Zamansky: Ann. Sci. Ecole Normale Sup., **66**, 19–93 (1949).
[6] A. Zygmund: Bull. Amer. Math. Soc., **51**, 274–278 (1945).
[7] A. Zygmund: Trigonometrical Series, Warszawa (1935).