108. *Hukuhara's Problem for Hyperbolic Equations with Two Independent Variables. II. Quasi-linear Case*

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1. **Introduction.** In Part I of this report, we have explained the concept of Hukuhara's problem (we shall use the abbreviation "Problem H" hereafter) for partial differential equations, and proved its correct posedness for the semi-linear hyperbolic systems with two independent variables. In this part, we show the same results for the quasi-linear system.

Consider the real quasi-linear system of the form

(1) $\partial u_i/\partial t-\lambda_i\ (t, x, u)\cdot\partial u_i/\partial x=f_i\ (t, x, u),\ \ i=1, 2, \cdots, N,$

where u stands for the N-dimensional vector $(u_1, \cdots, u_N)$. We adopt the notations λ, f, etc. to represent the vectors $(\lambda_1, \cdots, \lambda_N)$, $(f_1, \cdots, f_N)$, etc. We define the norm $\|g\|_D$ of a vector $g=(g_1, \cdots, g_N)$ whose components g_i are functions defined on some domain D, such as $\|g\|_D=\sup_i [\sup_D |g_i|]$.

Our Problem H for the quasi-linear system (1) is defined quite similarly to the semi-linear case. Namely, let N curves C_i, $i=1, 2, \cdots, N$ be given on a strip $G_0=\{0\leq t\leq B_0, B_0>0\}$ of (t, x)-space and let the value of the i-th unknown u_i of (1) be prescribed on the i-th curve C_i for $i=1, 2, \cdots, N$. Under these conditions we shall study the equations (1).

If we impose certain restrictions on the magnitude of the constant B_0 and on the situation of the curves C_i, then the Problem H has a solution which is unique and stable under a certain class of smooth functions.

1° We assume that the components of λ and f are defined and continuous on a strip $\overline{G}=\{0\leq t\leq B, \|u\|\leq\rho;\ B, \rho>0\}$ of (t, x, u)-space, where $\|u\|$ is defined such as $\|u\|=\operatorname{Sup}_i |u_i|$. We assume further that they have continuous derivatives up to second order with respect to u, x and their mixed differentiation, and that the norms $\|\cdot\|_{\overline{G}}$ of all those derived vectors including λ and f themselves are finite.

2° Conditions for the curves C_i

We require that for every i, the i-th curve C_i should be twice continuously differentiable, the absolute value of its curvature should be less than a constant Γ, and C_i should be *uniformly transversal* to the whole family of the i-th characteristics of (1) for every u such as $\|u\|\leq\rho$. The explanation of the last statement is as follows. Let

(t_0, x_0) be any point of (t, x)-space such as $0 \leq t_0 \leq B_0$. At this point the i-th characteristic $l_i(u)$ of (1) has the direction coefficient $-\lambda_i(t_0, x_0, u(t_0, x_0))$, so that the set of direction coefficients $\{-\lambda_i(t_0, x_0, u)\}_{\|u\| \leq \rho}$ gives us the *all possible directions* of the i-th characteristic at this point. Our requirement is that the minor angle between the direction of the i-th curve C_i and that of any possible direction of the i-th characteristic $l_i(u)$ should be bounded from below by a positive constant θ. This requirement is satisfied by all curves *nearly* parallel to x-axis since we have assumed the finiteness of $\|\lambda\|_{\bar{G}}$ in 1°.

3° Conditions for the prescribed values of the solution

We adopt the notation $\phi_i(t,x)$ to represent the prescribed value of the i-th unknown of (1), so that $\phi_i(t,x) = u_i(t,x)$ for all points of C_i, $i = 1, 2, \cdots, N$. We assume that for every i, ϕ_i is twice continuously differentiable along C_i, and that the inequalities $|\phi_i| < \rho$, $|\partial\phi_i/\partial s_i| \leq M_1$, $|\partial^2\phi_i/\partial s_i^2| \leq M_2$ hold with some constants M_1 and M_2, where $\partial/\partial s_i$ means the differentiation along C_i.

We define the *size* of the Problem H for the quasi-linear system (1) as the set of constants $B, \rho, \Gamma, \theta, M_1, M_2$ and the norms $\|\cdot\|_{\bar{G}}$ of the derived vectors of f and λ mentioned in 1°. Under these assumptions 1°, 2°, 3°, a solution of the Problem H for the system (1) exists on the strip $G_0 = \{0 \leq t \leq B_0\}$ of (t, x)-space, where B_0 is determined by several inequalities which consist of the quantities derived from the size of the problem. The special choice of curves C_i and functions ϕ_i does not affect the magnitude of B_0 except their contributions to the size of the problem.

If we require that the norm of the derivative of the solution with respect to x should be less than a constant K, then the solution of the Problem H is unique and stable. More precisely, the breadth B_0 of the domain G_0 is determined by the inequalities involving the quantities derived from the constant K and the size of the problem, and the solution of the problem such as $\|u\|_{G_0} \leq \rho$ and $\|\partial u/\partial x\|_{G_0} \leq K$, is unique and stable on G_0.

2. Existence of the solution. In this section we show the existence of the solution of the Problem H for the system (1) under the assumptions that the conditions 1° 2° 3° of § 1 are satisfied by it. To simplify the notations, we assume that f_i and λ_i are the functions of u only. The other cases can be treated quite similarly.

For the moment we impose on B_0 merely the condition such as $B_0 \leq B$ and proceed by formal calculations. Further conditions for B_0 will be stated later.

We construct the sequence $u^{(n)}$ successively by the sequence of the Problem H defined such as

$$(2) \qquad \partial u_i^{(n+1)}/\partial t - \lambda_i(u^{(n)}) \cdot \partial u_i^{(n+1)}/\partial x = f_i(u^{(n)}), \; u_i^{(n+1)} = \phi_i \qquad \text{on } C_i,$$

where we put $u_i^{(-1)}\equiv 0$ for convenience sake. If we integrate the equations (2) along their characteristics $l_i^{(n)}$ which pass through the point (t_0, x_0) and are expressed such as $x=\psi_i^{(n)}(t)$, then we obtain

$$(3)\qquad u_i^{(n+1)}(t_0, x_0)=\phi_i(t_i^{(n)}, x_i^{(n)})+\int_{t_i^{(n)}}^{t_0} f_i(u^{(n)}(t, \psi_i^{(n)}(t)))dt,$$

where $(t_i^{(n)}, x_i^{(n)})$ means the intersecting point of $l_i^{(n)}$ and C_i. Equations (2) are semi-linear (indeed linear) in the unknowns $u_i^{(n+1)}$, so that we can apply the theory of Part I and obtain for every n the unique solution $u^{(n+1)}$ on a strip $G_0=\{0\leq t\leq B_0\}$, where B_0 can be determined by the size of our quasi-linear problem independently of n. Every $u^{(n+1)}$ is twice continuously differentiable with respect to x and satisfies the integral equations (3). If the constant B_0 satisfies further inequalities, then $u^{(n)}$ converges uniformly on G_0 to a limit u which gives the solution of our problem. We shall explain the process briefly.

1° $\|u^{(n)}\|_{G_0}\leq\rho$ for all n.

2° $\|\partial u^{(n)}/\partial x_0\|_{G_0}\leq K$ for some constant K.

To prove it, the inequalities $|\partial\psi_i^{(n)}(t)/\partial x_0|\leq\exp(B_0\cdot M\cdot K)$ for $0\leq t\leq B_0$, are needed and proved by induction, where M is a constant derived from the size of the problem. The condition to be satisfied by B_0 is the form $M'\cdot\exp(B_0\cdot M\cdot K)+B_0\cdot M''\cdot\exp(B_0\cdot M\cdot K)\cdot K\leq K$, where M', M'' are the constants derived from the size and K must be selected to satisfy the inequalities such as $K>M'$ and $K\geq\|u^{(0)}\|_{G_0}$.

Similarly the boundedness of $\|\partial^2 u^{(n)}/\partial x_0^2\|_{G_0}$ is obtained under the appropriate condition for B_0.

3° $u^{(n)}$ converges uniformly on G_0 to a continuous function u as $n\to\infty$.

Subtracting side by side from (2) the similar equations with n replaced with m, we have

$$(4)\qquad \begin{aligned}&\frac{\partial}{\partial t}[u_i^{(n+1)}-u_i^{(m+1)}]-\lambda_i(u^{(n)})\cdot\frac{\partial}{\partial x}[u_i^{(n+1)}-u_i^{(m+1)}]\\ &=\frac{\partial u_i^{(m+1)}}{\partial x}\cdot[\lambda_i(u^{(n)})-\lambda_i(u^{(m)})]+f_i(u^{(n)})-f_i(u^{(m)}).\end{aligned}$$

As λ and f are Lipschitzian with some constants L and L' respectively, we have from (4) the inequality such as $\|u^{(n+1)}-u^{(m+1)}\|_{G_0}\leq(K\cdot L+L')\cdot B_0\cdot\|u^{(n)}-u^{(m)}\|_{G_0}$, which reduces as $m, n\to\infty$ to the formula $d\leq(K\cdot L+L')\cdot B_0\cdot d$, where $d=\limsup_{n,m\to\infty}\|u^{(n)}-u^{(m)}\|_{G_0}$ and K is the constant mentioned in 2°. If we impose on B_0 the condition $(K\cdot L+L')B_0<1$, then we must have $d=0$, which assures us of the uniform convergence of $u^{(n)}$ to a continuous limit function u. Consequently $\psi_i^{(n)}(t)$ and $(t_i^{(n)}, x_i^{(n)})$ of (3) converge uniformly on G_0 to a limit $\psi_i(t)$ and (t_i, x_i), and u gives the solution of the integral equations such as

$$u_i(t_0, x_0)=\phi_i(t_i, x_i)+\int_{t_i}^{t_0} f_i(u(t, \psi_i(t)))dt. \tag{5}$$

If we can prove the smoothness of the solution u of (5), it will give the solution of our Problem H. The i-th component u_i of u is continuously differentiable in the direction of the curve expressed such as $x=\psi_i(t)$, whose direction coefficient is $-\lambda_i(u(t, x))=\lim_{n\to\infty} -\lambda_i(u^{(n)}(t, x))$, so that to prove the smoothness of u_i, it remains only to examine the differentiation in x-direction.

4° $\partial u^{(n)}/\partial x_0$ converges uniformly on G_0 as $n\to\infty$.

Differentiating (4) with respect to x_0 and noticing that on the curve C_i the value of $[\partial u_i^{(n)}/\partial x_0-\partial u_i^{(m)}/\partial x_0]$ converges uniformly to 0 as $m, n\to\infty$, we can prove the inequality $d\leq B_0\cdot M\cdot d$, where

$$d=\limsup_{m,n\to\infty} \| \partial u^{(n)}/\partial x_0-\partial u^{(m)}/\partial x_0 \|_{G_0}$$

and M is a constant derived from the size of the problem. If we impose on B_0 the condition $B_0\cdot M<1$, then we must have $d=0$, and $\partial u^{(n)}/\partial x_0$ converges uniformly on G_0. Consequently, the limit u of $u^{(n)}$ is continuously differentiable with respect to x, so that u belongs to class C^1 and gives the solution of the Problem H for the quasi-linear system (1).

3. Uniqueness and stability of the solution. The solution u of the Problem H for the system (1) is *unique* and *stable* on a certain strip $G_0=\{0\leq t\leq B_0\}$ of (t, x)-space, if we require that u satisfies $\| u \|_{G_0}\leq\rho$ and $\| \partial u/\partial x \|_{G_0}\leq K$ with a constant K. We notice that the solution obtained in § 2 has these properties.

1° Uniqueness

Indeed, for any two such solutions u and v we should have

$$\frac{\partial}{\partial t}[u_i-v_i]-\lambda_i(u)\cdot\frac{\partial}{\partial x}[u_i-v_i]=\frac{\partial v_i}{\partial x}[\lambda_i(u)-\lambda_i(v)] \tag{6}$$
$$+f_i(u)-f_i(v).$$

Since λ and f are Lipschitzian with constants L and L' respectively, integrating (6) along its characteristics, we can prove the inequality $\| u-v \|_{G_0}\leq(K\cdot L+L')\cdot B_0\cdot\| u-v \|_{G_0}$, so that if we impose on B_0 the condition such as $(K\cdot L+L')\cdot B_0<1$, we must have $u=v$ on G_0.

2° Stability

Let two sets of data C_i, ϕ_i and $\overline{C_i}$, $\overline{\phi}_i$ be given whose contributions to the size are given by the same constants Γ, θ, M_1 and M_2. We assume that these data have the following properties. To any point (t_i, x_i) of C_i, we let correspond a point $(\overline{t_i}, \overline{x_i})$ of $\overline{C_i}$ which is the intersecting point of $\overline{C_i}$ and one of the *possible characteristics* $l_i(u)$ through the point (t_i, x_i). Namely, $l_i(u)$ is expressed as $x=\psi_i(t)$ by the solution

$\psi_i(t)$ of the ordinary differential equation such as $d\psi_i(t)/dt=-\lambda_i(u(t, \psi_i(t)))$, $\psi_i(t_i)=x_i$, where u is any smooth function defined on $G=\{0\leq t\leq B\}$ and satisfies the condition $\|u\|_G\leq\rho$. We require that the inequality $|t_i-\bar{t}_i|\leq\varepsilon$ should hold with a positive constant ε for every point of C_i and for every u such as is stated above. We assume further that functions ϕ_i and $\bar{\phi}_i$ are given on C_i and $\overline{C}_i$ in such a way that the inequality $|\phi_i(t_i, x_i)-\bar{\phi}_i(\bar{t}_i, \bar{x}_i)|\leq\delta$ should hold with a positive constant δ for every point of C_i and for every possible i-th characteristic. Under these assumptions we can prove easily

$$(7)\qquad \|u-\bar{u}\|_{G_0}\leq\delta+B_0\cdot L'\cdot(\|u-\bar{u}\|_{G_0}+K\cdot\|\psi-\bar{\psi}\|_{G_0})+\varepsilon\cdot\|f\|_{G_0},$$

where u and $\bar{u}$ are the solutions of the Problem H corresponding to the data C_i, ϕ_i and $\overline{C}_i$, $\bar{\phi}_i$. Since the order of the magnitude of $\|\psi-\bar{\psi}\|_{G_0}$ is that of $\|u-\bar{u}\|_{G_0}$, formula (7) assures us of the uniform convergence of $\bar{u}$ to u on a certain strip G_0 as δ, $\varepsilon\to 0$, which means the stability property of the solution of our problem.

Remarks. Under certain conditions, we can replace the infinite strip G with the finite domain G'. Definitions and arguments stated in the last section of Part I concerning semi-linear systems, are valid for quasi-linear systems with the single modification that we should consider *all possible characteristics* in this case. For example; if all N curves C_i pass through a fixed point (t_0, x_0) of G and if there exists a piecewise smooth curve l_0 such as $x=\psi_0(t)$, $0\leq t\leq B$, which satisfies the following conditions; $x_0=\psi_0(t_0)$, $d\psi_0(t)/dt\leq-\|\lambda\|_{\bar{G}}$ for $t>t_0$ and $d\psi_0(t)/dt\geq-\|\lambda\|_{\bar{G}}$ for $t<t_0$, then the *right finiteness condition* is satisfied along this l_0, where $G=\{0\leq t\leq B\}$ means the strip in (t, x)-space and the relation between G' and $\overline{G}'$ is similar to that between G and $\overline{G}$.

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