6. Convergence Concepts in Semi-ordered Linear Spaces. I

By Hidegorô NAKANO and Masahumi SASAKI

(Comm. by K. KUNUGI, M.J.A., Jan. 12, 1959)

Concerning semi-ordered linear spaces, L. Kantorovitch [1] gave originally two different concepts of convergence, that is, order convergence and star convergence. One of the authors introduced two other concepts, that is, dilatator convergence in [2] and individual convergence in [3], which are essentially equivalent to each other. Combining these concepts, we also obtain star-individual convergence in [4]. In this paper we want to discuss these concepts of convergence and their combinations more systematically. In the sequel we will use the terminologies and notations in the book [4].

Let R be a continuous semi-ordered linear space. We consider the order convergence basic, that is, for a sequence $a_\nu \in R$ ($\nu=0, 1, 2, \cdots$), $a_0=\lim_{\nu\to\infty} a_\nu$ means

$$a_0 = \bigcap_{\nu=1}^{\infty} \bigcup_{\mu\geq\nu} a_\mu = \bigcup_{\nu=1}^{\infty} \bigcap_{\mu\geq\nu} a_\mu.$$

In the sequel we denote by $\{a_\nu\}_\nu$ an arbitrary sequence $a_\nu \in R$ ($\nu=0, 1, 2,\cdots$) and $\{a_\nu\}_{\nu\geqq 1}$ means a_ν ($\nu=1, 2,\cdots$). A mapping $\mathfrak{a}$ of all sequences $\{a_\nu\}_\nu$ to sequences $\{a_\nu^{\mathfrak{a}}\}_\nu$ is called an *operator*, if

1) $\qquad a_0=\lim_{\nu\to\infty} a_\nu$ implies $a_0^{\mathfrak{a}}=\lim_{\nu\to\infty} a_\nu^{\mathfrak{a}}$,

2) $\qquad \{a_\nu^{\mathfrak{a}}\}_{\nu\geqq 1}$ depends only upon $\{a_\nu\}_{\nu\geqq 1}$

that is, $a_\nu=b_\nu$ ($\nu=1, 2,\cdots$) implies $a_\nu^{\mathfrak{a}}=b_\nu^{\mathfrak{a}}$ ($\nu=1, 2,\cdots$). An operator $\mathfrak{a}$ is said to be *linear* if

$$(\alpha a_\nu+\beta b_\nu)^{\mathfrak{a}}=\alpha a_\nu^{\mathfrak{a}}+\beta b_\nu^{\mathfrak{a}} \qquad (\nu=0, 1, 2,\cdots).$$

For two operators $\mathfrak{a}, \mathfrak{b}$, putting

$$a_\nu^{\mathfrak{a}\mathfrak{b}} = (a_\nu^{\mathfrak{a}})^{\mathfrak{b}} \qquad (\nu=0, 1, 2,\cdots),$$

we also obtain an operator $\mathfrak{a}\mathfrak{b}$, which will be called the *product* of $\mathfrak{a}$ and $\mathfrak{b}$. With this definition, we have obviously

$$(\mathfrak{a}\mathfrak{b})\mathfrak{c} = \mathfrak{a}(\mathfrak{b}\mathfrak{c}).$$

$\mathfrak{a}$ is said to *commute* $\mathfrak{b}$, if $\mathfrak{a}\mathfrak{b}=\mathfrak{b}\mathfrak{a}$.

A set $\mathfrak{A}$ of operators is called a *process*, if for any two sequences $\{a_\nu\}_\nu$, $\{b_\nu\}_\nu$ with $a_0 \neq b_0$ we can find $\mathfrak{a}\in\mathfrak{A}$ for which $a_0^{\mathfrak{a}}\neq b_0^{\mathfrak{a}}$. A set A of processes is called a *modificator*, if for any $\mathfrak{A}_1, \mathfrak{A}_2 \in A$ we can find $\mathfrak{A}\in A$ for which $\mathfrak{A}\subset\mathfrak{A}_1, \mathfrak{A}_2$. For two modificators A, B we write $A\geqq B$, if for any $\mathfrak{A}\in A$ we can find $\mathfrak{B}\in B$ for which $\mathfrak{A}\supset\mathfrak{B}$. If $A\geqq B$ and $B\geqq A$ at the same time, we write $A=B$.

Let A and B be modificators. For a process $\mathfrak{A}\in A$ and a system of processes $\mathfrak{B}_{\mathfrak{a}}\in B$ ($\mathfrak{a}\in\mathfrak{A}$) we see easily that the set

$$\{\mathfrak{a}\mathfrak{b}:\ \mathfrak{a}\in\mathfrak{A},\ \mathfrak{b}\in\mathfrak{B}_{\mathfrak{a}}\}$$

also is a process, and furthermore that all such processes constitute a modificator, which will be called the *product* of A and B, and denoted by AB. We also see that the system

$$\{\mathfrak{a}\mathfrak{b}:\ \mathfrak{a}\in\mathfrak{A},\ \mathfrak{b}\in\mathfrak{B}\}\qquad(\mathfrak{A}\in A,\ \mathfrak{B}\in B)$$

is a modificator, which will be called the *direct product* of A and B and denoted by $A\circ B$.

For modificators A, B, C we have obviously by definition

(1) $$(AB)C=A(BC),\quad (A\circ B)\circ C=A\circ(B\circ C),$$

(2) $$A\circ B\geqq AB,$$

(3) $A\geqq B$ implies $AC\geqq BC,\ CA\geqq CB,\ A\circ C\geqq B\circ C,\ C\circ A\geqq C\circ B,$

(4) $$(AB)\circ C\geqq A(B\circ C),\quad A\circ(BC)\geqq(A\circ B)C.$$

For a modificator A, a sequence $\{a_\nu\}_{\nu\geqq 1}$ is said to be *A-convergent*, if we can find $a_0\in R$ and $\mathfrak{A}\in A$ such that

$$a_0^{\mathfrak{a}}=\lim_{\nu\to\infty}a_\nu^{\mathfrak{a}}\quad\text{for all } \mathfrak{a}\in\mathfrak{A}.$$

In this case we see easily that such a_0 is determined uniquely. Thus such a_0 is called the *A-limit* of $\{a_\nu\}_{\nu\geqq 1}$ and we write

$$a_0=A\text{-}\lim_{\nu\to\infty}a_\nu.$$

With this definition we have obviously

Theorem 1. *For two modifications A, B we have*

$$a_0=AB\text{-}\lim_{\nu\to\infty}a_\nu$$

if and only if we can find $\mathfrak{A}\in A$ such that

$$a_0^{\mathfrak{a}}=B\text{-}\lim_{\nu\to\infty}a_\nu^{\mathfrak{a}}\quad\textit{for all } \mathfrak{a}\in\mathfrak{A}.$$

For two modificators A, B, we write $A\succ B$ if

$$a_0=A\text{-}\lim_{\nu\to\infty}a_\nu\ \text{ implies }\ a_0=B\text{-}\lim_{\nu\to\infty}a_\nu;$$

and A is said to be *equivalent* to B and denoted by $A\sim B$, if $A\succ B$ and $B\succ A$ at the same time. With this definition we see easily

(5) $A\geqq B$ implies $A\succ B,$

(6) $A\succ B$ implies $CA\succ CB,\ C\circ A\succ C\circ B,$

(7) $$A\succ A\circ B\succ AB.$$

A modificator A is said to *commute* an operator $\mathfrak{a}$, if

$$a_0=A\text{-}\lim_{\nu\to\infty}a_\nu\ \text{ implies }\ a_0^{\mathfrak{a}}=A\text{-}\lim_{\nu\to\infty}a_\nu^{\mathfrak{a}}.$$

With this definition we conclude immediately by Theorem 1

Theorem 2. *For two modificators A, B, if every operator of A commutes an operator $\mathfrak{c}$ and B commutes $\mathfrak{c}$, then AB commutes $\mathfrak{c}$.*

As the simplest operator we have the *identity* 1, that is, $a_\nu^1=a_\nu$ ($\nu=0,1,2,\cdots$). The modificator, which consists of only one process $\{1\}$, is denoted by O. O-convergence coincides obviously with the order convergence, that is, $a_0=O\text{-}\lim_{\nu\to\infty}a_\nu$ if and only if $a_0=\lim_{\nu\to\infty}a_\nu$. Furthermore we have for every modificator A

$$O\succ A,\quad OA=AO=O\circ A=A\circ O=A.$$

For every subsequence $\{\mu_\nu\}_{\nu\geqq 1}$ of $\{1, 2, \cdots\}$, putting

$$a_0^{\mathfrak{s}} = a_0, \quad a_\nu^{\mathfrak{s}} = a_{\mu_\nu} \quad (\nu = 1, 2, \cdots),$$

we obtain an operator $\mathfrak{s}$, which will be called a *sub. operator* and denoted by $\mathfrak{s}\{\mu_\nu\}$, if we need to indicate $\{\mu_\nu\}$. For two sub. operators $\mathfrak{s}_1, \mathfrak{s}_2$, the product $\mathfrak{s}_1\mathfrak{s}_2$ also is a sub. operator. We write $\mathfrak{s}\{\mu_\nu\} \geqq \mathfrak{s}\{\rho_\nu\}$ if $\{\rho_\nu\}$ is a subsequence of $\{\mu_\nu\}$.

We denote by S the modificator, which consists of all such processes $\mathfrak{S}$ of sub. operators that

1) $\mathfrak{s} \leqq \mathfrak{s}_0 \in \mathfrak{S}$ implies $\mathfrak{s} \in \mathfrak{S}$,

2) for any sub. operator $\mathfrak{s}$ we can find $\mathfrak{s}_0 \in \mathfrak{S}$ for which $\mathfrak{s} \geqq \mathfrak{s}_0$.

With this definition we have obviously

$$(8) \qquad SS = S \circ S = S.$$

For every projector $[p]$, putting $a_\nu^{\mathfrak{l}} = [p]a_\nu$ $(\nu = 0, 1, 2, \cdots)$, we obtain an operator $\mathfrak{l}$, which will be called a *loc. operator* and denoted by $\mathfrak{l}[p]$, if we need to indicate $[p]$. We write $\mathfrak{l}[p] \geqq \mathfrak{l}[q]$, if $[p] \geqq [q]$. We have obviously $\mathfrak{l}[p]\mathfrak{l}[q] = \mathfrak{l}[p][q]$ and $\mathfrak{l}\mathfrak{s} = \mathfrak{s}\mathfrak{l}$ for every loc. operator $\mathfrak{l}$ and sub. operator $\mathfrak{s}$.

We denote by L the modificator which consists of all such processes $\mathfrak{L}$ of loc. operators that

1) $\mathfrak{l} \leqq \mathfrak{l}_0 \in \mathfrak{L}$ implies $\mathfrak{l} \in \mathfrak{L}$,

2) for any loc. operator $\mathfrak{l}$ we can find $\mathfrak{l}_0 \in \mathfrak{L}$ for which $\mathfrak{l} \geqq \mathfrak{l}_0$.

With this definition we have obviously

$$(9) \qquad LL = L \circ L = L.$$

Since $\mathfrak{s}\mathfrak{l} = \mathfrak{l}\mathfrak{s}$ for every loc. operator $\mathfrak{l}$ and sub. operator $\mathfrak{s}$, we have

$$(10) \qquad L \circ S = S \circ L.$$

Lemma 1. *Let A be a modificator, which commutes every loc. operator. In order that*

$$a_0 = LA\text{-}\lim_{\nu\to\infty} a_\nu,$$

it is necessary and sufficient that we can find a system of projectors $[p_\lambda]$ $(\lambda \in \Lambda)$ such that

$$\bigcup_{\lambda\in\Lambda} [p_\lambda] \bigcup_{\nu=1}^{\infty} [a_\nu] = \bigcup_{\nu=1}^{\infty} [a_\nu]$$

$$[p_\lambda]a_0 = A\text{-}\lim_{\nu\to\infty} [p_\lambda]a_\nu \quad \textit{for all } \lambda \in \Lambda.$$

Proof. We need only to prove the sufficiency. For such a system of projectors $[p_\lambda]$ $(\lambda \in \Lambda)$, denoting by $\mathfrak{L}$ the set of all such $\mathfrak{l}[p]$ that $[p] \leqq [p_\lambda]$ for some $\lambda \in \Lambda$ or $[p][p_\lambda] = 0$ for all $\lambda \in \Lambda$, we see easily that $\mathfrak{L} \in L$, and

$$a_0^{\mathfrak{l}} = A\text{-}\lim_{\nu\to\infty} a_\nu^{\mathfrak{l}} \quad \text{for all } \mathfrak{l} \in \mathfrak{L},$$

because A commutes $\mathfrak{l}$ by assumption.

For two elements $p \geqq 0 \geqq q$ in R, putting

$$a_\nu^{\mathfrak{i}} = (a_\nu \frown p) \smile q \quad (\nu = 0, 1, 2, \cdots),$$

we obtain an operator $\mathfrak{i}$, which will be called an *ind. operator* and denoted by $\mathfrak{i}(p, q)$ if we need to indicate p, q. We write $\mathfrak{i}(p, q)\geqq\mathfrak{i}(r, s)$ if $p\geqq r\geqq s\geqq q$. We have obviously

$$\mathfrak{i}(p, q)\mathfrak{i}(r, s)=\mathfrak{i}(p\frown r, q\smile s)$$

and $\mathfrak{i}\mathfrak{l}=\mathfrak{l}\mathfrak{i}$, $\mathfrak{i}\mathfrak{s}=\mathfrak{s}\mathfrak{i}$ for every loc. operator $\mathfrak{l}$ and sub. operator $\mathfrak{s}$.

We denote by I the modificator which consists of only one process of all ind. operators. With this definition we have obviously

(11) $$II=I\circ I=I.$$

From the proof of Theorem 1.1 in [3], we conclude easily

(12) $$I\sim L.$$

Lemma 2. *In order that* $a_0=I\text{-}\lim_{\nu\to\infty} a_\nu$, *it is necessary and sufficient that we can find a sequence* $0\leqq p_1\leqq p_2\leqq\cdots$ *such that*

$$(a_0\frown p_\mu)\smile(-p_\mu)=\lim_{\nu\to\infty}(a_\nu\frown p_\mu)\smile(-p_\mu) \quad \textit{for all } \mu=1, 2,\cdots,$$

$$\lim_{\mu\to\infty}(x\frown p_\mu)\smile(-p_\mu)=x \quad \textit{for all } x\in[a_1, a_2,\cdots]R.$$

Proof. We need only to prove the sufficiency. Putting $\mathfrak{i}_\mu=\mathfrak{i}(p_\mu, -p_\mu)$ $(\mu=1, 2,\cdots)$, we obtain by assumption for any ind. operator $\mathfrak{i}$

$$(\overline{\lim_{\nu\to\infty}}\, a_\nu^{\mathfrak{i}})^{\mathfrak{i}_\mu}=\overline{\lim_{\nu\to\infty}}\, a_\nu^{\mathfrak{i}\mathfrak{i}_\mu}=(\overline{\lim_{\nu\to\infty}}\, a_\nu^{\mathfrak{i}_\mu})^{\mathfrak{i}}=a_0^{\mathfrak{i}_\mu\mathfrak{i}}=(a_0^{\mathfrak{i}})^{\mathfrak{i}_\mu}$$

Thus, making $\mu\to\infty$, we obtain $\overline{\lim_{\nu\to\infty}}\, a_\nu^{\mathfrak{i}}=a_0^{\mathfrak{i}}$. We conclude similarly also that $\underline{\lim_{\nu\to\infty}}\, a_\nu^{\mathfrak{i}}=a_0^{\mathfrak{i}}$. Therefore $a_0=I\text{-}\lim_{\nu\to\infty} a_\nu$ by definition.

As $\mathfrak{i}\mathfrak{l}=\mathfrak{l}\mathfrak{i}$ and I consists of only one process, we have by definition

(13) $$I\circ L = L\circ I = LI.$$

Recalling (12), we obtain by (9), (11)

(14) $$LI\sim IL\sim I.$$

As $\mathfrak{i}\mathfrak{s}=\mathfrak{s}\mathfrak{i}$, we have

(15) $$I\circ S = S\circ I = SI.$$

As $I\circ S\geqq IS$ by (2), we have hence $SI\succ IS$ by (5). Now we shall prove

(16) $$SI\sim IS.$$

We suppose $a_0=IS\text{-}\lim_{\nu\to\infty} a_\nu$. Putting $p_\mu=\mu\sum_{\nu=1}^{\mu}|a_\nu|$ $(\mu=1, 2,\cdots)$, we see easily that the sequence $0\leqq p_1\leqq p_2\leqq\cdots$ satisfies the condition of Lemma 2. For any sub. operator $\mathfrak{s}$, we can find by assumption a sequence of operators $\mathfrak{s}\geqq\mathfrak{s}_1\geqq\mathfrak{s}_2\geqq\cdots$ such that

$$(a_0\frown p_\mu)\smile(-p_\mu)=\lim_{\nu\to\infty}(a_\nu^{\mathfrak{s}_\mu}\frown p_\mu)\smile(-p_\mu) \quad (\mu=1, 2,\cdots).$$

Then we can find by the diagonal method a sub. operator $\mathfrak{s}_0\leqq\mathfrak{s}$ such that

$$(a_0\frown p_\mu)\smile(-p_\mu)=\lim_{\nu\to\infty}(a_\nu^{\mathfrak{s}_0}\frown p_\mu)\smile(-p_\mu) \quad (\mu=1, 2,\cdots).$$

Thus we have $a_0=SI\text{-}\lim_{\nu\to\infty} a_\nu$, and therefore $IS\succ SI$ by definition.

A modificator is said to be *regular*, if it commutes every sub., loc. and ind. operators. The modificator O is obviously regular.

Lemma 3. *If a modificator A is regular, then all SA, LA and IA are regular, and $S\circ A\prec A$, $L\circ A\prec A$, $I\circ A\prec A$.*

Proof. By virtue of Theorem 2, both LA and IA are regular. To prove that SA is regular, we need only to show that SA commutes every sub. operator. We suppose that $a_0=SA\text{-}\lim_{\nu\to\infty} a_\nu$. Then we can find by Theorem 1 a process $\mathfrak{S}\in S$ such that

$$a_0^{\mathfrak{s}} = A\text{-}\lim_{\nu\to\infty} a_\nu^{\mathfrak{s}} \quad \text{for all } \mathfrak{s}\in\mathfrak{S}.$$

For any sub. operator $\mathfrak{s}_0$, we obtain hence

$$a_0^{\mathfrak{s}_0\mathfrak{s}} = A\text{-}\lim_{\nu\to\infty} a_\nu^{\mathfrak{s}_0\mathfrak{s}} \quad \text{for } \mathfrak{s}_0\mathfrak{s}\in\mathfrak{S}.$$

Putting $\mathfrak{S}_0=\{\mathfrak{s}\colon \mathfrak{s}_0\mathfrak{s}\in\mathfrak{S}\}$, we see easily that $\mathfrak{S}_0\in S$. Thus we have $a_0^{\mathfrak{s}_0}=SA\text{-}\lim_{\nu\to\infty} a_\nu^{\mathfrak{s}_0}$. Therefore SA commutes every sub. operator. If A is regular, then we have obviously $S\circ A\prec A$, $L\circ A\prec A$, $I\circ A\prec A$ by definition.

Lemma 4. *If R is super-universally continuous and a modificator A commutes every loc. operator, then we have*

$$(L\circ S)A\sim LSA\succ SLA.$$

Proof. We suppose that $a_0=LSA\text{-}\lim_{\nu\to\infty} a_\nu$. As R is super-universally continuous by assumption, we can find $[p_\mu]$ $(\mu=1,2,\cdots)$ such that

$$[p_\mu]a_0=SA\text{-}\lim_{\nu\to\infty}[p_\mu]a_\nu \ (\mu=1,2,\cdots), \quad \bigcup_{\mu=1}^{\infty}[p_\mu]\geqq\bigcup_{\nu=1}^{\infty}[a_\nu].$$

Then we can find $\mathfrak{S}_\mu\in S$ by definition such that

$$[p_\mu]a_0^{\mathfrak{s}}=A\text{-}\lim_{\nu\to\infty}[p_\mu]a_\nu^{\mathfrak{s}} \quad \text{for all } \mathfrak{s}\in\mathfrak{S}_\mu \ (\mu=1,2,\cdots).$$

Denoting by $\mathfrak{S}$ the intersection of all $\mathfrak{S}_\mu$ $(\mu=1,2,\cdots)$, we see easily by the diagonal method that $\mathfrak{S}\in S$. Denoting by $\mathfrak{L}$ the set of all $\mathfrak{l}[p]$ such that $[p]\leqq[p_\mu]$ for some $\mu=1,2,\cdots$ or $[p][p_\mu]=0$ for all $\mu=1,2,\cdots$, we see easily that $\mathfrak{L}\in L$, because A commutes every loc. operator by assumption. Thus we have

$$a_0^{\mathfrak{l}\mathfrak{s}}=A\text{-}\lim_{\nu\to\infty} a_\nu^{\mathfrak{l}\mathfrak{s}} \quad \text{for all } \mathfrak{l}\in\mathfrak{L},\ \mathfrak{s}\in\mathfrak{S},$$

and hence $a_0=(L\circ S)A\text{-}\lim_{\nu\to\infty} a_\nu$. Therefore we have $LSA\succ(L\circ S)A$. On the other hand we have $(L\circ S)A\succ LSA$ by (2), (3). Consequently $(L\circ S)A\sim LSA$. As $L\circ S=S\circ L\geqq SL$, we obtain hence $LSA\succ SLA$.

A modificator is said to be *standard*, if it is composed only of O, S, L, I by the product and the direct product.

Theorem 3. *If R is super-universally continuous, then every standard modificator is equivalent to one of O, S, L, LS, SL.*

Proof. We need only to show $SLS\sim LSL\sim ILS\sim ISL\sim SL$. As $LS\succ SL$ by Lemma 4, we obtain by (6), (8), (7): $SLS\succ SSL=SL\succ SLS$, and by (9), Lemma 3: $LSL\succ SLL=SL\succ LSL$. As $L\sim I$ by (12), we have by (6), (16), (11): $ISL\sim ISI\sim IIS=IS\sim SI\sim SL$. As $IL\leqq LI$ by

(13), (2), we have by (3), (16), (12): $ILS \leqq LIS \sim LSI \sim LSL \sim SL$. On the other hand we have $ILS \succ ISL$ by Lemma 4 and (6), and $ISL \sim SL$, as proved just above.

Theorem 4. *If R is super-universally continuous and complete,*[*] *then every standard modificator is equivalent to one of O and S.*

Proof. If R is super-universally continuous and complete, then we see easily $I \sim L \sim O$. Thus we obtain by Lemmas 3 and 4

$$S \succ LS \succ SL \sim SO = S.$$

Therefore we conclude our assertion from Theorem 3.

References

[1] L. Kantorovitch: Lineare halbgeordnete Räume, Math. Sbornik, **2** (44), 121–168 (1937).

[2] H. Nakano: Teilweise geordnete Algebra, Jap. Jour. Math., **17**, 485–511 (1941).

[3] ——: Ergodic theorems in semi-ordered linear spaces, Ann. Math., **49**, 538–556 (1948).

[4] ——: Modulared Semi-ordered Linear Spaces, Tokyo (1950).

[5] ——: Modern Spectral Theory, Tokyo (1950).

*) A semi-ordered linear space is said to be *complete* if every orthogonal sequence of elements is bounded (cf. [5]).