4. On a Theorem on Modular Lattices

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1. It is well known that an irreducible, complete, (upper and lower) continuous, complemented modular lattice L is finite-dimensional if and only if the following condition is satisfied:[1]

Condition Δ. *L contains no infinite sequence (a_i) of nonzero elements a_i, $i=1,2,\cdots$, such that for every $i>1$ there exists an element b_i satisfying $a_{i-1}\geq a_i \dot{\cup} b_i$[2] and $a_i \approx b_i$.*

The purpose of the present paper is to prove the following theorem. By $m(L)$ we denote the least upper bound of all integers r such that L contains an independent system of mutually projective nonzero r elements.

Theorem. For any complete upper continuous modular lattice L the condition Δ is equivalent to each of the following two conditions:

Condition M. *$m(L)$ is finite.*

Condition F. *There is no independent countable subset (a_i) such that $a_i \gtrsim a_{i+1} \neq 0$ for every i.*[3]

As a consequence of this we shall obtain

Corollary 1. *Let $\mathfrak{R}$ be a semisimple ring with unit element and assume that $\mathfrak{R}$-left (-right) module $\mathfrak{R}$ is injective. Then $\mathfrak{R}$ is a regular ring (in the sense of v. Neumann), and the following three conditions are equivalent:*

(i) *$\mathfrak{R}$ is of bounded index.*

(ii) *$\mathfrak{R}/\mathfrak{P}$ is a simple ring with minimum condition for every primitive ideal $\mathfrak{P}$.*

(iii) *$\mathfrak{R}$ is P-soluble.*[4]

In this case, $\mathfrak{R}$-right (-left) module $\mathfrak{R}$ is also injective.

2. Henceforth L always will denote a modular lattice with zero.

Lemma 1. Let $a\cap b=a\cap c=0$ and $a\cup b\geq c$. Then $(a\cup c)\cap b\sim_a c$.[5]

Lemma 2. If $0\neq a\leq b=b_1\dot{\cup} b_2 \dot{\cup}\cdots\dot{\cup} b_n$, then there exist nonzero a', b' such that $a\geq a'\sim b'\leq b_i$ for some i.

In fact, if $a\cap(b_2\cup\cdots\cup b_n)=0$, then $b_1\cap(a\cup b_2\cup\cdots\cup b_n)\sim a$ by Lemma 1; hence Lemma 2 follows by induction.

1) See [7].

2) $\dot{\cup}$ denotes the join of independent elements.

3) By $a\gtrsim b$ we mean the existence of c such that $a\geq c\approx b$.

4) See [5].

5) $b\sim_a c$ is meant that $a\dot{\cup} b=a\dot{\cup} c$.

We denote $m(L(0, a))$ for the interval $L(0, a)$ of L by $m(a)$.

Lemma 3. If $a \cap b = 0$, then $m(a \cup b) \leq m(a) + m(b)$.

Proof. If either $m(a)$ or $m(b)$ is ∞ or 0, Lemma is obvious. Let $0 < m(a), m(b) < \infty$. Suppose $a \dot\cup b \geq x_1 \dot\cup \cdots \dot\cup x_u$ where $u = m(a) + m(b) + 1$ and $x_i \approx x_j \neq 0$ for every i, j. If, say, $x_1 \cap b \neq 0$, then we replace x_1 by $x_1 \cap b$ and each of the other x_i by a suitable element which is contained in it and projective to $x_1 \cap b$. Repeating this process we may assume without loss of generality that $x_i \cap b = 0$ or $x_i \leq b$ for every i. Let $x_j \cap b = 0$, $j = 1, \cdots, r$, and $x_k \leq b$, $k = r+1, \cdots, u$. If $r = 0$, then $u \leq m(b)$, a contradiction. Hence $r > 0$. Set $(x_j \cup b) \cap a \equiv x'_j$, $j = 1, \cdots, r$. By Lemma 1, $x'_j \approx x_j$. If $\perp(x'_1, \cdots, x'_r)$ we have a contradiction since it follows from this that $\perp(x'_1, \cdots, x'_r, x_{r+1}, \cdots, x_u)$ and $u \leq m(a) + m(b)$. Thus, $\perp(x'_1, \cdots, x'_p)$ and not $\perp(x'_1, \cdots, x'_{p+1})$ for some p. Since $b \dot\cup x_j = b \dot\cup x'_j$ by Lemma 1, if $\perp(b, x_1, \cdots, x_{p+1})$ we see that $\perp(b, x'_1, \cdots, x'_{p+1})$ and $\perp(x'_1, \cdots, x'_{p+1})$ which is a contradiction. Hence $f \equiv b \cap (x_1 \cup \cdots \cup x_{p+1}) \neq 0$. By Lemma 2 there exist mutually projective nonzero $\bar{f}, \bar{x}_j$, $j = 1, \cdots, p+1$ such that $\bar{f} \leq f$ and $\bar{x}_j \leq x_j$. Let $\bar{x}_k$, $k = p+2, \cdots, u$, be elements satisfying $\bar{x}_k \leq x_k$ and $\bar{x}_k \approx \bar{f}$. Clearly $\perp(x'_1, \cdots, x'_p, b)$, and so $\perp(x_1, \cdots, x_p, b)$, whence $\perp(\bar{x}_1, \cdots, \bar{x}_p, \bar{f})$. Since $\bar{x}_1 \cup \cdots \cup \bar{x}_p \cup \bar{f} \leq x_1 \cup \cdots \cup x_{p+1}$, it follows that $\perp(\bar{x}_1, \cdots, \bar{x}_p, \bar{f}, \bar{x}_{p+2}, \cdots, \bar{x}_u)$. Therefore we have obtained an independent system of mutually projective u elements in which $u - r + 1$ elements are contained in b. Repeating this procedure we may arrive at the case that $m(b) + 1$ of x_i are contained in b, and have a contradiction as desired.

For any element $a \in L$ we denote by a^* the set of all elements x with the properties that (i) $a \geq x$ and (ii) if $a \geq y \neq 0$ then $x \cap y \neq 0$. Thus, if $a^* \ni b$ for some $a \neq 0$, then $b \neq 0$. $a^* \ni b$ and $b^* \ni c$ imply $a^* \ni c$. $a^* \ni b$ and $a \geq c$ mean $c^* \ni b \cap c$. Hence $a^* \ni b \cap c$ if $a^* \ni b, c$.

An element a is called an *m-element* provided that (i) $a \geq b$, $a \cap c = 0$ and $b \approx c$ imply $b = c = 0$; (ii) there are mutually projective elements $a_1, \cdots, a_n$ such that $a = a_1 \dot\cup \cdots \dot\cup a_n$ and $m(a_i) = 1$, $i = 1, \cdots, n$. In this case it follows from Lemma 3 that $m(a) = n$.

Lemma 4. Let $0 \neq b \leq a_1 \dot\cup \cdots \dot\cup a_n$ and let every a_i be an m-element. Then, $b \cap a_i \neq 0$ for some i.

Proof is easily obtained from Lemma 2 and the definition of m-elements.

Lemma 5. Let a be an m-element such that $m(a) < n$. If $b = b_1 \dot\cup \cdots \dot\cup b_n$ and $b_i \approx b_j$ for every i, j, then $a \cap b = 0$.

Proof. Let $a \cap b \neq 0$. By Lemma 2, $a \geq a' \approx b' \leq b_j$ for some $a' \neq 0$, b' and j. Denote the projective isomorphism between $L(0, b_j)$ and $L(0, b_i)$ by T_i, $i = 1, \cdots, n$. Suppose that $b'T_i \geq x$ and $a \cap x = a \cap b'T_i \cap x = 0$.

Then $x\approx xT_i^{-1}\leq b'\approx a'\leq a$, so that $x=0$, whence $a\cap b'T_i\in(b'T_i)^*$ and so $(a\cap b'T_i)T_i^{-1}\in b'^*$. It follows from this that $b''\equiv\bigcap_{i=1}^n(a\cap b'T_i)T_i^{-1}\in b'^*$. Clearly $b''\neq 0$ since $b'\neq 0$. Now $b''T_i\leq a\cap b'T_i\leq a\cap b_i$ and we have $a\geq b''T_1\dot{\cup}b''T_2\dot{\cup}\cdots\dot{\cup}b''T_n\neq 0$. This implies that $m(a)\geq n$ and yields a contradiction.

Lemma 6. Let a_i, $i=1,2,\cdots$, be a finite or infinite sequence of m-elements. If $m(a_i)\neq m(a_j)$ for every $i\neq j$, then (a_i) is an independent system.[6)]

Proof. With no loss of generality we may suppose that $m(a_i)<m(a_{i+1})$ for $i=1,2,\cdots$. Let $(a_1\dot{\cup}\cdots\dot{\cup}a_n)\cap a_{n+1}\neq 0$. By Lemma 4, $a_i\cap a_{n+1}\neq 0$ for some $1\leq i\leq n$. This contradicts Lemma 5. Therefore, $\perp(a_1,\cdots,a_{n+1})$ and $\perp(a_i, i=1,2,\cdots)$ by induction.

Lemma 7. Assume that L satisfies the condition F. If $a_1,\cdots,a_n$ are mutually projective and independent elements such that $m(a_i)=1$ for every i, then there exists an m-element b with the properties that $m(b)\geq n$ and $(a_1\cup\cdots\cup a_n)\cap b\neq 0$.

Proof. Let us suppose that Lemma is false. Now we shall construct an infinite set of elements x_{ij}, $i=n,n+1,\cdots$, $j=1,\cdots,i$, satisfying the conditions that (i) x_{ij}, $j=1,\cdots,i$, are mutually projective and independent, (ii) $x_{i+1,j}\leq x_{ij}$, (iii) $m(x_{ij})=1$ for every i,j, and (iv) x_{ii}, $i=n,n+1,\cdots$, are independent. First we set $a_j\equiv x_{nj}$, $j=1,\cdots,n$. Assume that we have constructed x_{ij}, $i=n,\cdots,n'$, $j=1,\cdots,i$, with the properties above. By Lemma 3 $m(x_{n'1}\cup\ldots\cup x_{n'n'})=n'\geq n$. Since $a_j=x_{nj}\geq x_{n'j}$, $j=1,\cdots,n$, we have $(a_1\cup\cdots\cup a_n)\cap(x_{n'1}\cup\cdots\cup x_{n'n'})\geq x_{n'1}\cup\cdots\cup x_{n'n}\neq 0$. From these we see that $x_{n1}\dot{\cup}\cdots\dot{\cup}x_{n'n'}$ is not an m-element. Hence $x_{n'1}\cup\cdots\cup x_{n'n'}\geq y$, $(x_{n'1}\cup\cdots\cup x_{n'n'})\cap z=0$ and $y\approx z\neq 0$ for some y,z. By Lemma 2 and the projectivities between $x_{n'j}$, there are mutually projective nonzero elements $x_{n'+1,j}$, $j=1,\cdots,n'+1$, such that $x_{n'+1,j}\leq x_{n'j}$, $j=1,\cdots,n'$, and $x_{n'+1,n'+1}\leq z$. Evidently $(x_{n'1}\cup\cdots\cup x_{n'n'})\cap x_{n'+1,n'+1}=0$ and $x_{n'+1,j}$, $j=1,\cdots,n'+1$, are independent. Now put $d_{n'+1}\equiv(\dot{\bigcup}_{i=n}^{n'}x_{ii})\cap x_{n'+1,n'+1}$. Then each $x_{n'+1,j}$, $j=1,\cdots,n'$, contains an element d_j projective to $d_{n'+1}$. Evidently, d_j, $j=1,\cdots,n'+1$, are independent and $d_j\leq x_{n'+1,j}\leq x_{jj}$, $j=n,\cdots,n'$. Hence $\dot{\bigcup}_{j=n}^{n'+1}d_j\leq\dot{\bigcup}_{i=n}^{n'}x_{ii}$. Since $m(\dot{\bigcup}_{j=n}^{n'}x_{jj})\leq\sum_{j=n}^{n'}m(x_{jj})=n'-n$ by Lemma 3, this implies that $d_{n'+1}=0$, and hence x_{ii}, $i=n,\cdots,n'+1$, are independent, as desired.

Now $x_{i+1,i+1}\approx x_{i+1,i}\leq x_{ii}$, i.e. $x_{ii}\gtrsim x_{i+1,i+1}$. By virtue of the independence of x_{ii}, $i=n,n+1,\cdots$, we have a contradiction to F, completing the proof.

Proof of Theorem. $(F\Rightarrow\Delta)$ Let (a_i) and (b_i) be infinite sequences such that $a_i\geq a_{i+1}\dot{\cup}b_{i+1}$ and $a_{i+1}\approx b_{i+1}$ for every i. Then $b_{i+1}\leq a_i\approx b_i$ and

6) An infinite set of elements of L is said to be independent in case every finite subset is independent.

$b_i \gtrsim b_{i+1}$. Since clearly b_i $(i>1)$ are independent, this contradicts F. ($\varDelta$ and $F \Rightarrow M$) First we note that the values $m(a)$ for all m-elements a are bounded. In fact, if not we may find an infinite sequence (a_i) of m-elements such that $m(a_{i+1})>m(a_i)$. Then (a_i) is independent by Lemma 6. Let $a_n=\dot{\bigcup}_{j=1}^{m(a_n)} a_{nj}$ and $a_{nj}\approx a_{nj'}$ for every j, j'. Set $b_j \equiv \bigcup_{n=j}^{\infty} a_{nj}$. Clearly $b_j \succ b_{j+1}$ and (b_j) is indepedent, contradicting F. Thus, $m(a)$ for m-elements a are bounded. Denote its maximum by m. Let $L\ni \dot{\bigcup}_{i=1}^{k} a_i$ and assume $a_i\approx a_{i'}\neq 0$ for every i, i'. It follows easily from $\varDelta$ that a_1 contains an element a_1' such that $m(a_1')=1$. If we replace a_1 by a_1' and each of the other a_i by an element contained in it and projective to a_1', we may assume with loss of generality $m(a_1)=\cdots=m(a_k)=1$. From Lemma 7 there exists an m-element b such that $m(b)\geq k$. Therefore $m\geq k$ and $m(L)=m$. ($M\Rightarrow F$) Let (a_i) be a countable independent set such that $a_i \gtrsim a_{i+1}$ for every i. Then, for every n there are mutually projective nonzero b_{ni}, $i=1,\cdots,n$ satisfying $b_{ni}\leq a_i$. Since b_{ni} are independent we get $m(L)=\infty$, contradicting M. ($\varDelta \Rightarrow F$) Let (a_i) be a countable independent set such that $a_i \gtrsim a_{i+1}$ for every i. Then for every n there exist mutually projective nonzero $b_{nj}, j=1,\cdots,2^n$ satisfying $b_{nj}\leq a_{2^n+j-1}$. Put $c_t\equiv \bigcup_{n=t}^{\infty}\bigcup_{k=1}^{2n-t} b_{nk}$ and $c_t'\equiv \bigcup_{n=t}^{\infty}\bigcup_{k=2n-t+1}^{2n-t+1} b_{nk}$. Then $c_t\approx c_t'$ and $c_{t-1}\geq c_t \dot{\bigcup} c_t'$, which contradicts $\varDelta$.

3. Let $\mathfrak{R}$ be a semisimple I-ring, and $L_{\mathfrak{R}}$ the lattice of all left (right) ideals of $\mathfrak{R}$. In a recent paper we have noted that $m(L_{\mathfrak{R}})$ coincides with the index of $\mathfrak{R}$.[7] Therefore, as an immediate consequence of our Theorem we obtain

Corollary 2. *Let $\mathfrak{R}$ be a semisimple I-ring. Then the following conditions are equivalent*:

(a) *$\mathfrak{R}$ is of bounded index.*

(b) *There is no infinite sequence of nonzero left (right) ideals $\mathfrak{l}_i$ such that $\mathfrak{l}_i \supsetneqq \mathfrak{l}_{i+1}\oplus\mathfrak{l}_{i+1}'$, $\mathfrak{l}_{i+1}'$ being a left (right) ideal isomorphic to $\mathfrak{l}_{i+1}$.*

(c) *There is no infinite sequence of nonzero left (right) ideals $\mathfrak{l}_i$ such that the sum $\sum \mathfrak{l}_i$ is direct and $\mathfrak{l}_{i+1}$ is isomorphic to a subideal of $\mathfrak{l}_i$.*

For any module $\mathfrak{M}$ we denote by $\mathfrak{M}^*$ the set of all submodules $\mathfrak{N}$ of $\mathfrak{M}$ with the property that $\mathfrak{N}\cap\mathfrak{N}'\neq 0$ for every submodule $\mathfrak{N}'\neq 0$ of $\mathfrak{M}$.

Lemma 8. Let $\mathfrak{Q}$ be the minimal injective extension[8] of a module $\mathfrak{M}$, and $\mathfrak{E}$ the endomorphism ring of $\mathfrak{Q}$. Then the radical of $\mathfrak{E}$ is the set N of all endomorphisms θ of $\mathfrak{Q}$ satisfying Ker $\theta\in\mathfrak{Q}^*$. Moreover, $\mathfrak{E}/N$ is isomorphic to the extended centralizer over $\mathfrak{M}$ and hence is regular. In case $\mathfrak{E}$ is semisimple, every submodule $\mathfrak{N}$ of $\mathfrak{M}$ has the

7) See [9, Lemma 4].

8) See [1, Section 4].

unique minimal injective extension $\overline{\mathfrak{N}}$ contained in $\mathfrak{O}$. $\overline{\mathfrak{N}}$ is the sum of all essential extensions of $\mathfrak{N}$ in $\mathfrak{O}$.

Proof. If Ker $\theta\in\mathfrak{O}^*$ for $\theta\in\mathfrak{E}$, then Ker $(1+\theta)=0$, since Ker $\theta\cap$ Ker$(1+\theta)=0$. Hence Im$(1+\theta)$ $(\simeq\mathfrak{O})$ is a direct summand of $\mathfrak{O}$ and clearly contains Ker θ $(\in\mathfrak{O}^*)$; this implies Im$(1+\theta)=\mathfrak{O}$ and θ is quasi-regular in $\mathfrak{E}$. It is easy to prove that N is a two-sided ideal of $\mathfrak{E}$ and that $\mathfrak{E}/N$ is isomorphic to the extended centralizer[9] over $\mathfrak{M}$. Since any extended centralizer is regular, $\mathfrak{E}/N$ is semisimple and N is the radical of $\mathfrak{E}$. Next, assume that $\mathfrak{E}$ is semisimple, and let $\overline{\mathfrak{N}}$ and $\mathfrak{N}'$ be minimal injective extensions of $\mathfrak{N}$ in $\mathfrak{O}$. By $\mathfrak{N}^c$ we denote a maximal submodule disjoint to $\mathfrak{N}$. Since $\overline{\mathfrak{N}}$ is an essential extension of $\mathfrak{N}$,[10] we have $\overline{\mathfrak{N}}\cap\mathfrak{N}^c=0$. Now, there is an element $\theta\in\mathfrak{E}$ such that $\overline{\mathfrak{N}}^\theta=\mathfrak{N}'$ and $(\mathfrak{N}\oplus\mathfrak{N}^c)(1-\theta)=0$. Clearly $\mathfrak{N}\oplus\mathfrak{N}^c\in\mathfrak{O}^*$, and so $1-\theta\in\mathfrak{N}$, hence $1=\theta$. Therefore $\mathfrak{N}'=\overline{\mathfrak{N}}$.

Proof of Corollary 1. By Lemma 8, $\mathfrak{R}$ is a regular ring. We denote the lattice of all principal left ideals of $\mathfrak{R}$ by $\bar{L}_{\mathfrak{R}}$. Let $(\mathfrak{R}e_\alpha)\in\bar{L}_{\mathfrak{R}}$. Then the minimal injective extension left ideal of $\sum\mathfrak{R}e_\alpha$ is uniquely determined by Lemma 8, and is clearly the join $\bigcup\mathfrak{R}e_\alpha$ of $(\mathfrak{R}e_\alpha)$. Hence $\bar{L}_{\mathfrak{R}}$ is complete. To see the upper continuity of $\bar{L}_{\mathfrak{R}}$ we assume that $(\mathfrak{R}e_\alpha)$ is simply ordered. Since $\bigcup\mathfrak{R}e_\alpha$ is an essential extension of $\sum\mathfrak{R}e_\alpha$, that is, $(\bigcup\mathfrak{R}e_\alpha)^*\ni\sum\mathfrak{R}e_\alpha$, we have $((\bigcup\mathfrak{R}e_\alpha)\cap\mathfrak{R}f)^*\ni(\sum\mathfrak{R}e_\alpha)\cap\mathfrak{R}f=\sum(\mathfrak{R}e_\alpha\cap\mathfrak{R}f)$ for any $\mathfrak{R}f\in\bar{L}_{\mathfrak{R}}$. Hence $(\bigcup\mathfrak{R}e_\alpha)\cap\mathfrak{R}f\subseteq\bigcup(\mathfrak{R}e_\alpha\cap\mathfrak{R}f)$, which shows the upper continuity of $\bar{L}_{\mathfrak{R}}$. Thus, it follows from Theorem that for $\bar{L}_{\mathfrak{R}}$ Conditions $\varDelta$ and M are equivalent. Now, (i)$\Rightarrow$(ii)$\Rightarrow$(iii) are known.[11] Levitzki proved that an FI-ring is P-soluble if and only if it satisfies the D-condition.[12] It is not too hard to see that the D-condition for a regular ring $\mathfrak{R}$ is equivalent to Condition $\varDelta$ for $\bar{L}_{\mathfrak{R}}$. On the other hand, the boundedness of indeces in $\mathfrak{R}$ is equivalent to Condition M for $L_{\mathfrak{R}}$, and hence also to Condition M for $\bar{L}_{\mathfrak{R}}$. Therefore we have (iii)$\Leftrightarrow$(i). The last statement of Corollary 1 follows from [8, Theorem 5], completing the proof.

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9) See [3].

10) See [1, (4.1)].

11) See [4, Theorems 5.6 and 5.7].

12) See [5] and [6, Corollary 1 of Theorem 5.3].

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