22. *An Abstract Analyticity in Time for Solutions of a Diffusion Equation*

By Kôsaku YOSIDA

Department of Mathematics, University of Tokyo

(Comm. by Z. SUETUNA, M.J.A., March 12, 1959)

1. *Introduction and the result.* Consider an equation of evolution

(1.1) $$\frac{\partial u}{\partial t}=Au, \quad t>0,$$

where the differential operator

(1.2) $$A=a^{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j}+b^i(x)\frac{\partial}{\partial x_i}+c(x)$$

is elliptic in a connected domain G of an m-dimensional euclidean space E^m. Under certain conditions upon the coefficients a, b and c of A, we can specify a linear subspace D of $L_2(G)$ with the following three properties.

(i) The functions $\in D$ are C^∞ in G, and D is $L_2(G)$-dense in $L_2(G)$ such that $Af \in L_2(G)$ for $f \in D$.

(ii) If we consider A as an operator on $D \subseteqq L_2(G)$ into $L_2(G)$, then A admits, in $L_2(G)$, the smallest closed extension $\hat{A}$.

(iii) $\hat{A}$ is the infinitesimal generator of a semi-group T_t of normal type in $L_2(G)$ such that, for any $f \in L_2(G)$, $u(t, x)=(T_t f)(x)$ is a solution of (1.1) with the initial condition

(1.1)′ $$L_2(G)\text{-}\lim_{t\downarrow 0} u(t, x)=f(x)$$

satisfying the "forward and backward unique continuation property":

(1.3) If, for a fixed $t_0>0$, $u(t_0, x)\equiv 0$ on an open set $G_0 \subseteqq G$, then $u(t, x)=0$ for every $t>0$ and every $x \in G_0$.

The proof of (1.3) is based upon the fact that $T_t f$ is an $L_2(G)$-valued abstract analytic function of t in a certain sector of the complex plane which contains the positive t-axis in its interior and with $t=0$ as its vertex. Such abstract analyticity in time is implied by the estimate (2.11) below of the resolvent of $\hat{A}$.[1]

Our result (1.3) gives a partial answer to a conjecture proposed by S. Ito and H. Yamabe [2]. Actually, our solution $u(t, x)=(T_t f)(x)$ enjoys the "unique continuation property":

(1.3)′ If, for a fixed $t_0>0$, $u(t_0, x)\equiv 0$ on an open set $G_0 \subseteqq G$, then $u(t, x)=0$ for every $t>0$ and every $x \in G$.

1) This estimate was given in the author's lecture at Yale University in the fall of 1958.

This may be proved by combining (1.3) with the "space-like unique continuation theorem for solutions of parabolic equations" obtained recently by S. Mizohata [3]. Thus we obtain another proof of the unique continuation theorem of S. Ito and H. Yamabe [2].[2)]

2. *The proof of the result.* For the sake of simplicity of exposition, we shall be concerned with the case[3)] $G=E^m$. We assume that the real-valued coefficients a, b and c are C^∞ in E^m and that

(2.1) $a^{ij}(x)$ and its first and second partials, $b^i(x)$ and its first partials and $c(x)$ are, in absolute values, all bounded on E^m by a positive constant β.

Thus the strict ellipticity of A implies the existence of two positive constants γ and δ such that

$$\gamma \sum_{j=1}^{m} \xi_j^2 \geqq a^{ij}(x)\xi_i\xi_j \geqq \delta \sum_{j=1}^{m} \xi_j^2 \qquad \text{on } E^m \tag{2.2}$$

for any real vector $(\xi_1, \xi_2, \cdots, \xi_m)$.

Let $H_1=H_1(E^m)$ be the space of complex-valued C^∞ functions $f(x)=f(x_1, \cdots, x_m)$ in E^m for which

$$||f||_1=\Big(\int_{E^m} |f(x)|^2 dx+\sum_{j=1}^{m} \int_{E^m} |f_{x_j}(x)|^2 dx\Big)^{1/2}<\infty, \tag{2.3}$$

and let $\widehat{H}_1=L_2(E^m)=L_2$ be the completion of H_1 with respect to the norm

$$||f||=\Big(\int_{E^m} |f(x)|^2 dx\Big)^{1/2} \tag{2.4}$$

We denote by RH_1 (and RL_2) the totality of real-valued functions belonging to H_1 (and to L_2).

Lemma. There exist two positive constants α_0 and β_0 such that, for any $f \in RH_1$, the equation

$$\alpha u - Au = f, \quad \alpha > \max(\alpha_0, \delta+\beta_0), \tag{2.5}$$

admits a uniquely determined solution $u(x)=u_f(x) \in RH_1$, and we have the estimate

$$||u_f|| \leqq (\alpha-\delta-\beta_0)^{-1} ||f||. \tag{2.6}$$

Proof. The existence of the solution $u_f \in RH_1$ for sufficiently large α is proved in K. Yosida [4]. If we denote by (f, g) the inner product $\int_{E^m} f(x)\overline{g(x)}dx$, then for any $u \in RH_1$,

$$||(\alpha I-A)u|| \cdot ||u|| \geqq |((\alpha I-A)u, u)| \tag{2.7}$$

by Schwarz inequality. By partial integration, we have (see K. Yosida [4])

2) For, these two authors treat the case where $\hat{A}$ is self-adjoint with its spectrum lying on negative real axis, and the estimate (2.11) is clear for such operator $\hat{A}$.

3) If G is a bounded domain of E^m, the method of the following proof may be modified so as to apply to the case where A is an elliptic differential operator of $2n$-order $(n>1)$.

$$(2.8)\quad \begin{aligned}((\alpha I-A)u, u)=\alpha\|u\|^2+\int_{E^m} a^{ij}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}dx+\int_{E^m}\frac{\partial a^{ij}}{\partial x_i}\frac{\partial u}{\partial x_j}u\,dx \\ -\int_{E^m} b^i\frac{\partial u}{\partial x_i}u\,dx-\int_{E^m} c\,u\,u\,dx.\end{aligned}$$

Hence we have, by (2.1)-(2.2) and the inequality $|\varepsilon\eta|\leqq 2^{-1}(|\varepsilon|^2+|\eta|^2)$,

$$(2.9)\quad \begin{aligned}((\alpha I-A)u, u)&\geqq\alpha\|u\|^2+\delta(\|u\|_1^2-\|u\|^2) \\ &\quad -m\beta[\nu(\|u\|_1^2-\|u\|^2)+\nu^{-1}m\|u\|^2+m^{-1}\|u\|^2] \\ &=[\alpha-\delta-m\beta(m\nu^{-1}-\nu+m^{-1})]\|u\|^2+(\delta-m\beta\nu)\|u\|_1^2\end{aligned}$$

for any $\nu>0$. Thus we have (2.6) from (2.7), by taking $\nu>0$ so small that $(\delta-m\beta\nu)>0$ and $\beta_0=m\beta(m\nu^{-1}-\nu+m^{-1})>0$.

Corollary. Let us consider A as an operator defined on $\{f; f\in RH_1, Af\in RH_1\}\subseteqq RL_2$ into RL_2. Then the smallest closed extension $\widetilde{A}$, in RL_2, of A satisfies the condition that, for $\alpha>\max(\alpha_0, \delta+\beta_0)$, the inverse $(\alpha I-\widetilde{A})^{-1}$ exists as a bounded linear operator defined on RL_2 into RL_2 with the estimate

$$(2.10)\quad \|(\alpha I-\widetilde{A})^{-1}\|\leqq(\alpha-\delta-\beta_0)^{-1}.$$

Theorem 1. If we consider A as an operator on $\{f; f\in H_1, Af\in H_1\}\subseteqq L_2$ into L_2, then the smallest closed extension $\hat{A}$, in L_2, of A is the infinitesimal generator of a semi-group T_t in L_2 which is strongly continuous in t, $\|T_t\|\leqq\exp((\delta+\beta_0)t)$ and such that

$$(2.11)\quad \overline{\lim_{|\tau|\uparrow\infty}}|\tau|\cdot\|((\alpha+\sqrt{-1}\tau)I-A)^{-1}\|<\infty.$$

Proof. By the lemma and the reality of the coefficients of A, we see that the range $(\alpha I-A)\cdot H_1$ is, for $\alpha>\max(\alpha_0, \delta+\beta_0)$, L_2-dense in L_2. Moreover we have, for $(u+\sqrt{-1}v)\in H_1$,

$$\begin{aligned}\|(\alpha I-A)(u+\sqrt{-1}v)\|^2&=\|(\alpha I-A)u\|^2+\|(\alpha I-A)v\|^2 \\ &\geqq(\alpha-\delta-\beta_0)^2\|u\|^2+(\alpha-\delta-\beta_0)^2\|v\|^2.\end{aligned}$$

Thus $(\alpha I-\hat{A})^{-1}$ is a bounded linear operator on L_2 into L_2 satisfying

$$(2.12)\quad \|(\alpha I-\hat{A})^{-1}\|\leqq(\alpha-\delta-\beta_0)^{-1}.$$

Hence the first part of the theorem is proved (see E. Hille-R. S. Phillips [1] or K. Yosida [5]). We have to show that (2.11) holds good. We have, for $w\in H_1$, $\alpha>\max(\alpha_0, \delta+\beta_0)$,

$$\|((\alpha+\sqrt{-1}\tau)I-A)w\|\cdot\|w\|\geqq|(((\alpha+\sqrt{-1}\tau)I-A)w, w)|.$$

As in (2.9), we obtain

$$\begin{aligned}&|\text{Real Part }(((\alpha+\sqrt{-1}\tau)I-A)w, w)| \\ &=\alpha\|w\|^2+\text{Real Part }\Big(\int_{E^m} a^{ij}\frac{\partial w}{\partial x_i}\frac{\partial \overline{w}}{\partial x_j}dx+\int_{E^m}\frac{\partial a^{ij}}{\partial x_i}\frac{\partial w}{\partial x_j}\overline{w}\,dx \\ &\qquad -\int_{E^m} b^i\frac{\partial w}{\partial x_i}\overline{w}\,dx-\int_{E^m} c\,w\,\overline{w}\,dx\Big)\Big| \\ &\geqq(\alpha-\delta-\beta_0)\|w\|^2+(\delta-m\beta\nu)\|w\|_1^2.\end{aligned}$$

Similarly we have

$$|\text{Imaginary Part } (((\alpha+\sqrt{-1}\tau)I-A)w, w)|$$
$$\geqq \big| |\tau|\cdot\|w\|^2 - m\beta\{\|w\|_1^2+m\|w\|^2\}\big| = \big|(|\tau|-m^2\beta)\|w\|^2 - m\beta\|w\|_1^2\big|.$$

If we assume that there exists $w\in H_1$, $\|w\|\neq 0$, such that

$$|\text{Imaginary Part } (((\alpha+\sqrt{-1}\tau)I-A)w, w)| \leqq 2^{-1}(|\tau|-m^2\beta)\|w\|^2$$

for sufficiently large τ (or for sufficiently large $-\tau$), then, for such large τ (or $-\tau$),

$$m\beta\|w\|_1^2 \geqq 2^{-1}(|\tau|-m^2\beta)\|w\|^2.$$

Hence, for such large τ (or $-\tau$),

$$|\text{Real Part } (((\alpha+\sqrt{-1}\,\tau)I-A)w, w)| \geqq (\delta-m\beta\nu)\frac{(|\tau|-m^2\beta)}{2m\beta}\|w\|^2.$$

Thus (2.11) is proved.

Theorem 2. The semi-group T_t is, for $t>0$, strongly differentiable in t any number of times. Actually, if we denote by $T_t^{(k)}$ the k-th strong derivative of T_t with respect to t, then there exists a positive constant ε such that, for any $t>0$, the sequence of operators

$$\sum_{k=0}^{n}(k!)^{-1}(\lambda-t)^k T_t^{(k)}$$

is, as $n\uparrow\infty$, convergent in the sense of the norm of operators when

(2.12) $$|\lambda-t|<\varepsilon t.$$

Proof. See K. Yosida [6].[4)]

Corollary. For any $f\in L_2$, $u(t,x)=(Tf)(x)$ is infinitely differentiable in $t>0$ and $x\in E^m$ and satisfies the Cauchy problem (1.1) - (1.1)$'$.

Proof. If we apply, in the sense of the distribution of L. Schwartz, the elliptic differential operator

$$\left(\frac{\partial^2}{\partial t^2}+A\right)$$

any number of times to $u(t,x)$, then the result is locally square integrable in the product space $(0<t<\infty)\times E^m$. Thus $u(t,x)$ is equivalent to a function which is C^∞ in $(0<t<\infty)\times E^m$. See, for the details, K. Yosida [4].

Proof of 1.3. Since $T_t^{(k)}=A^kT_t$, we have, by Theorem 2,

$$\lim_{n\to\infty}\|T_{t_0+h}f-\sum_{k=0}^{n}(k!)^{-1}h^kA^kT_{t_0}f\|=0$$

for sufficiently small h. Hence there exists a sequence $\{n'\}$ of natural numbers such that

$$u(t_0+h,x)=\lim_{n'\to\infty}\sum_{k=0}^{n'}(k!)^{-1}h^kA^ku(t_0,x)\qquad\text{for almost all } x\in E^m.$$

By the hypothesis in (1.3), we have $A^ku(t_0,x)\equiv 0$ in G_0, and hence $u(t_0+h,x)\equiv 0$ in G_0. Repeating the process we see that $u(t,x)=0$ for every $t>0$ and every $x\in G_0$.

4) The "if" part of Theorem 2 in K. Yosida [6] must be corrected as: if $\lim_{|\tau|\uparrow\infty}\log|\tau|\cdot\|R(1+i\tau,A)\|=0$, then T_t' exists for every $t>0$.

References

[1] E. Hille and R. S. Phillips: Functional Analysis and Semi-groups, New York (1958).

[2] S. Ito and H. Yamabe: A unique continuation theorem for solutions of a parabolic equations, J. Math. Soc. Japan, **10**, no. 3, 314–321 (1958).

[3] S. Mizohata: Unicité du prolongement des solutions pour quelques opérateurs différentiels paraboliques, Mem. Colleg. Sci. Univ. Kyoto, sect. A, **31**, no. 3, 219–239 (1958).

[4] K. Yosida: An operator-theoretical integration of wave equations, J. Math. Soc. Japan, **8**, no. 1, 79–92 (1956).

[5] K. Yosida: On Semi-group Theory and Its Application to Cauchy's Problem in Partial Differential Equations, Bombay (1957).

[6] K. Yosida: On the differentiability of semi-groups of linear operators, Proc. Japan Acad., **34**, 337–340 (1958).