51. On Extreme Elements in Lattices

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In a series of papers [2-6] we have studied the concept of *B-covers* and *B^*-covers* in lattices. $B(a, b)=\{x \mid axb\}$, $B^*(a,b)=\{y \mid aby\}$, where axb means that $x=(a\smile x)\frown(b\smile x)=(a\frown x)\smile(b\frown x)$. $B(a, b)$ is called the *B-cover* of a and b. We shall say that an element e of a lattice L is an *extreme element* to an element x of L (or e is *extreme* to x) if $B^*(x, e)=e$. An element e is called *extreme* if it is *extreme* to some element of L. By $(a, b)E$ we shall mean that b is *extreme* to a (that is, $B^*(a, b)=b$). We shall call (a, b) an *extreme pair* when $(a, b)E$ and $(b, a)E$; we denote it by $(a, b)E_s$.

If $(a, b)E_s$ and a and b are comparable, then (a, b) equals (O, I). Elements O, I satisfying OxI for all x are called "*extreme*" by G. Birkhoff [1]. If a and b are complemented, then $(a, b)E_s$ by our definition [Theorem 1]. In Theorem 2, we shall give a representation of a Boolean algebra by *maximal extreme B-covers*. If $(a, b)E$, then we shall be able to find out an *extreme pair* $(a_n, b)E_s$ by Theorem 4. If the space of a topological lattice is compact, then we shall call this space a *compact lattice*. After Birkhoff [1], a chain is complete if and only if it is topologically compact. If we denote by $E(a)$ the set of all elements which are extreme to an element a in a compact lattice, then we shall find some interesting properties of $E(a)$ [Theorems 7 and 8], and we shall prove that a *compact extreme lattice* which consists of *extreme elements* is a complemented lattice [Theorem 9].

Theorem 1. *If a and a' are complemented in a lattice, then $(a, a')E_s$.*

Proof. If $aa'x$, then we have $a'=(a\frown a')\smile(a'\frown x)=a'\frown x$, $a'=(a\smile a')\frown(a'\smile x)=a'\smile x$ from $a\frown a'=0$, $a\smile a'=I$, and hence we have $a'=x$, thus we have $B^*(a, a')=a'$. Similarly we have $B^*(a', a)=a$. Hence we have $(a, a')E_s$. The converse of this theorem is not always true.

Lemma 1. *$B(a, b)=B(a\frown b, a\smile b)$ in a distributive lattice.*

Proof. This is proved by [3, Theorem 3].

Lemma 2. *In a Boolean algebra L, if $(a, b)E_s$, then $a\smile b=I$, $a\frown b=O$.*

Proof. Let a' be the complement of a, then $B(a, a')=B(a\frown a', a\smile a')=B(O, I)=L$ by Lemma 1. Hence $b\in B(a, a')$, that is, aba'.

Accordingly if $(a, b)E_s$, then $b=a'$.

Definition. $B(a, b)$ is called a *maximal extreme B-cover* if $(a, b)E_s$ and if there exists no *extreme B-cover* $B(c, d)$ such that $B(c, d)\supset B(a, b)$ but neither $B(a, b)=B(c, d)$ nor $B(c, d)=L$.

Theorem 2. *In a Boolean algebra L, any extreme B-cover is a maximal extreme B-cover and* $E(L)=L=B(a, a')$, *where* $E(L)$ *is the set of extreme elements of L, and a' is the complement of a.*

Proof. This is proved by Theorem 1 and Lemma 2.

Lemma 3. *In a lattice axb implies* $a\smile b\geqq x\geqq a\frown b$.

Proof. Since $x=(a\smile x)\frown(b\smile x)\geqq x\smile(a\frown b)\geqq x$ we have $x\smile(a\frown b)=x$, hence $x\geqq a\frown b$. Similarly we have $a\smile b\geqq x$.

Lemma 4. *acb implies* $B^*(a, b)\subset B^*(c, b)$ *in a lattice.*

Proof. *acb* and *abx* imply *cbx* by [3, Lemma 4].

Lemma 5. $B^*(a, b)=B^*(a\smile b, b)\frown B^*(a\frown b, b)$ *in a lattice.*

Proof. Since $a\smile b\in B(a, b)$ we have $B^*(a, b)\subset B^*(a\smile b, b)$ by Lemma 4. Similarly $B^*(a, b)\subset B^*(a\frown b, b)$. Conversely if x belongs to $B^*(a\smile b, b)$ and $B^*(a\frown b, b)$, then $x\in B^*(a, b)$.

Lemma 6. *If* $x\in B^*(a, b)$, *then* $x\in B^*(a', b)$ *for any* a' *such that* $a\smile b\geqq a'\geqq a\frown b$.

Proof. $B^*(a', b)=B^*(a'\smile b, b)\frown B^*(a'\frown b, b)$, $B^*(a, b)=B^*(a\smile b, b)\frown B^*(a\frown b, b)$ by Lemma 5, but $B^*(a\smile b, b)\subset B^*(a'\smile b, b)$, $B^*(a\frown b, b)\subset B^*(a'\frown b, b)$ by Lemma 4; hence we have $B^*(a, b)\subset B^*(a', b)$.

Lemma 7. *If* $(a, b)E$, *then* $(c, b)E$ *for any c such that* $c\smile b\geqq a\smile b$, $c\frown b\leqq a\frown b$.

Proof. As in the proof of Lemma 6, we have $b\in B^*(c, b)\subset B^*(a, b)=b$, and hence $B^*(c, b)=b$, that is, $(c, b)E$.

Now we shall write $(a, b)E'$ when b is not extreme to a.

Theorem 3. *In any lattice*

(1) *if* $(a', b)E$, $(b', a)E$ *for* $a', b'\in B(a\frown b, a\smile b)$, *then we have* $(a, b)E_s$;

(2) *if b is not extreme for some c satisfying* $c\smile b\geqq a\smile b$, $c\frown b\leqq a\frown b$, *then* $(a, b)E'$.

Proof. (1) If $(a', b)E$ for $a'\in B(a\frown b, a\smile b)$, then we have $(a, b)E$ by Lemma 7, similarly we have $(b, a)E$. (2) is proved immediately from Lemma 6.

Theorem 4. *In a lattice if* $(a, b)E$ *and* $B^*(b, a)\ni a_1\neq a$, *then we have* $(a_1, b)E$. *Moreover if there exists* $a_2\neq a_1$ *such that* $B^*(b, a_1)\ni a_2$, *then* $(a_2, b)E$; *thus if we find, by repeating this method, an element* a_n *such that* $B^*(b, a_n)=a_n$, *then* $(a_n, b)E_s$.

Proof. If $B^*(b, a)\ni a_1$, then $b\smile a_1\geqq b\smile a$, $b\frown a_1\leqq b\frown a$ by Lemma 3 and hence $(a_1, b)E$ by Lemma 7. Similarly we have $(a_n, b)E$, and hence we have $(a_n, b)E_s$ together with $(b, a_n)E$.

Lemma 8. *For* $a\neq 0$, $E(a)\not\ni I$ *if and only if there exists* $x\neq I$

such that $a\smile x=I$.

Proof. If aIx, then $a\smile x\geqq I$ by Lemma 3, and hence we have $a\smile x=I$. If $a\smile x=I$, then we have aIx by the definition.

Lemma 9. *If* $(x,I)E$, *then* $(y,I)E$ *for* $y\leqq x$.

Proof. Suppose that $(y,I)E'$; then there exists u such that $y\smile u=I$, $u\neq I$ by Lemma 8, hence $x\smile u=I$ from $I=y\smile u\leqq x\smile u$, this contradicts the hypothesis.

Lemma 10. *If* $(x,a)E$ *and* $(y,a)E$, *then* $(z,a)E$ *for* $y\leqq z\leqq x$.

Proof. Suppose that $(z,a)E'$; then there exists $u\neq a$ satisfying zau, so that $a=(z\smile a)\frown(a\smile u)\geqq(y\smile a)\frown(a\smile u)\geqq a$ and $a=(z\frown a)\smile(a\frown u)\leqq(x\frown a)\smile(a\frown u)\leqq a$. Hence we have ① $(y\smile a)\frown(a\smile u)=a$ and ② $(x\frown a)\smile(a\frown u)=a$. In this case, (i) if $u\geqq a$, then we have yau together with ① and (ii) if $u\leqq a$, then we have xau together with ②, and (iii) when a and u are non-comparable, let $u_1=a\smile u$, $u_2=a\frown u$, then we have yau_1 and xau_2 since zau implies zau_1 and zau_2. In each case of (i), (ii), (iii), we have a contradiction to the hypothesis. Thus we have the assertion.

Theorem 5. *Let* $C=\{c\mid E(c)\ni O, I, a, b$, *where* $(a,b)E_s\}$ *in a lattice. If* $C\ni x,y$ *for* $x\geqq y$, *then we have* $z\in C$ *for* $x\geqq z\geqq y$.

Proof. This is a consequence of Lemmas 9 and 10.

Theorem 6. *In a lattice if* $(d,a)E$, $(e,b)E$ *and* $M=B^*(a,d)\frown B^*(b,e)$, *then* $E(x)\ni a,b$ *for* $x\in M$.

Proof. If $(d,a)E$ and $B^*(a,d)\ni d_1$, then we have $(d_1,a)E$ by Th. 4. Similarly if $(e,b)E$ and $B^*(b,e)\ni e_1$, then we have $(e_1,b)E$.

Henceforth we shall assume that L is a compact lattice with O and I.

Theorem 7. *In a compact lattice we have*

(1) $E(c)=I$ *if and only if* $c=O$,

(2) $E(c)=\{O,I\}$ *if and only if* $L=(c]\smile[c)$, *where* $[c)=\{z\mid z\geqq c\}$, $(c]=\{z\mid z\leqq c\}$.

Proof. (1) Suppose that $E(c)=I$ and $c\neq O$; then there exists a non-comparable element b_1 to c satisfying $c\frown b_1=O$ since $E(c)\not\ni O$ by the dual of Lemma 8. Since $(c,b_1)E'$ by the hypothesis there exists b_2 such that $B^*(c,b_1)\ni b_2\neq b_1$. From $(c\frown b_1)\smile(b_1\frown b_2)=b_1$ and $c\frown b_1=O$ we have $b_2>b_1$ and hence $b_2\smile c\geqq b_1\smile c$, but $b_2\smile c\neq b_1\smile c$, for if $b_2\smile c=b_1\smile c$, then $B^*(c,b_1)=B^*(c\smile b_1,b_1)\frown B^*(c\frown b_1,b_1)=B^*(c\smile b_2,b_1)\frown B^*(O,b_1)\ni b_2$ by Lemma 5, and hence $(c\smile b_2)b_1b_2$. On the other hand, $(c\smile b_2)b_2b_1$ from $b_1<b_2\leqq c\smile b_2$, thus we have $b_1=b_2$, a contradiction. Then we have $b_1\smile c<b_2\smile c$.

Similarly since $(c,b_2)E'$ there exists b_3 such that $B^*(c,b_2)\ni b_3\neq b_2$ and $c\smile b_2<c\smile b_3$. Accordingly we have an increasing chain $b_1<b_2<\cdots<b_n<\cdots$, and hence $c\smile b_1<c\smile b_2<\cdots<c\smile b_n<\cdots$. Since L is a compact lattice, we have $b_n\to b_0$, and hence $c\smile b_n\to c\smile b_0$.

Furthermore we have cb_1b_3, where b_3 is non-comparable to c, for $(c\smile b_1)\frown(b_1\smile b_3)=(c\smile b_1)\frown b_3=(c\smile b_1)\frown(c\smile b_2)\frown b_3=(c\smile b_1)\frown b_2=b_1$ by cb_1b_2, cb_2b_3. And if $b_3\geqq c$, then $(c\smile b_1)\frown b_3=c\smile b_1\neq b_1$ contrary to cb_1b_3, and if $c\geqq b_3$, then $c\geqq b_1$ contrary to the hypothesis, thus b_3 is non-comparable to c. Similarly we have cb_1b_n, where b_n is non-comparable to c, and cb_1b_n tends to cb_1b_0 as $b_n\to b_0$ since L is a compact lattice. Then, we have $b_1\smile(c\frown b_0)=b_1$, and hence b_0 is non-comparable to c. On the other hand, we have $(c, b_0)E$ from the meaning of least upper bound, this contradicts the hypothesis. Consequently we have $c=O$. The converse is trivial.

(2) We shall prove that there is no element which is non-comparable to c. Let b_1 be a non-comparable element to c.

Since $E(c)=\{O, I\}$ we have $c\frown b_1>O$, $c\smile b_1<I$ and $(c, b_1)E'$, hence there exists $d\neq b_1$ satisfying $B^*(c, b_1)\ni d$. If $d>b_1$, let $d\equiv b_2$ and if $d<b_1$, then let $b_2'\equiv d$. If d is non-comparable to b_1, then let $b_2\equiv b_1\smile d$, $b_2'\equiv b_1\frown d$. In these cases b_2 and b_2' are both non-comparable to c and $b_1\smile c<b_2\smile c$, $b_1\frown c<b_2'\frown c$ as in (1).

Repeating this method we have two chains, increasing and decreasing, as follows:

$b_1<b_2<\cdots<b_n<\cdots$; $b_1>b_2'>\cdots>b_n'>\cdots$, where b_n and b_n' are non-comparable to c and $cb_1b_2, cb_1b_3,\cdots, cb_1b_n,\cdots$ and $cb_1b_2', cb_1b_3',\cdots, cb_1b_n',\cdots$ (it may happen that one of those sequences does not occur).

Since L is a compact lattice $cb_1b_n\to cb_1b_0$ and $cb_1b_n'\to cb_1b_0'$ as $b_n\to b_0$ and $b_n'\to b_0'$ respectively, where b_0 and b_0' are non-comparable to c and $(c, b_0)E$, $(c, b_0')E$ in the same way as in (1). This is a contradiction, thus we have the assertion of (2).

Theorem 8. *In a compact lattice $E(a)=b$ implies $a\smile b=I$, $a\frown b=O$.*

Proof. Since it is obtained by (1) Th. 7 in case $b=I$, we may prove in case $b\neq O, I$, whence $a\neq O, I$. From $E(a)\bar{\ni} O, I$ there exists b_1, b_1' such that $a\smile b_1=I$, $a\frown b_1'=O$. When $a\frown b_1=O$ or $a\smile b_1'=I$ we have $b=b_1=b_1'$ satisfying $a\smile b=I$ and $a\frown b=O$ from $E(a)=b$ and Theorem 1.

If b_1, b_1' are both distinct from b, that is, $a\frown b_1>O$ and $a\smile b_1'<I$, then $B^*(a, b_1)\ni b_2$ such that $b_2<b_1$, $a\frown b_2<a\frown b_1$ and $B^*(a, b_1')\ni b_2'$ such that $b_2'>b_1'$, $a\smile b_2'>a\smile b_1'$ since $a\smile b_1=I$, $a\frown b_1'=O$ and $E(a)\bar{\ni} b_1, b_1'$. Moreover since $a\smile b_2\geqq a\smile b_1=I$ and $a\frown b_2'\leqq a\frown b_1'=O$ from $ab_1b_2, ab_1'b_2'$ by Lemma 3, we have $a\smile b_2=I$ and $a\frown b_2'=O$. If $a\frown b_2>O$ and $a\smile b_2'<I$, then repeating this method we have increasing and decreasing chains $\{b_n\}$ and $\{b_n'\}$, where

$a\frown b_1>a\frown b_2>\cdots>a\frown b_n>\cdots$; $a\smile b_1'<a\smile b_2'<\cdots<a\smile b_n'<\cdots$;
$a\smile b_1=a\smile b_2=\cdots=I$, $a\frown b_1'=a\frown b_2'=\cdots=O$.

If $b_n\to b_0$ and $b_n'\to b_0'$, then since L is a topological lattice we have

$E(a) \ni b_0, b_0'$ in the same way as in (2), Th. 7 and $a_0 \smile b_0 = I$, $a_0 \frown b_0' = O$. Thus we have $b_0 = b_0' = b$, satisfying $a \smile b = I$, $a \frown b = O$, this completes the proof.

Now we shall call a lattice L an *extreme lattice* when every element of L is *extreme*.

Lemma 11. *xya, xyb, $a \geqq c \geqq b$ imply xyc.*

Proof. By xya, xyb and $a \geqq c \geqq b$ we have

$$y = (x \smile y) \frown (y \smile b) \leqq (x \smile y) \frown (y \smile c) \leqq (x \smile y) \frown (y \smile a) = y,$$

$$y = (x \frown y) \smile (y \frown b) \leqq (x \frown y) \smile (y \frown c) \leqq (x \frown y) \smile (y \frown a) = y,$$

and hence we have xyc.

Lemma 12. *In case $a \geqq b$, $a \neq I$ and $b \neq O$, if there exists z such that $x \smile z = I$, $x \frown z = O$ for any $x \in B(a, b)$, then $\{B(a, b), z\}$ is an extreme lattice. In this case if there exists y such that $x_1 \smile z = y < I$ for some $x_1 \in B(a, b)$, then $\{B(a, b), z\}$ is not an extreme lattice.*

Proof. The first part of this theorem is obtained from Theorem 1. In the latter part, since $z \smile a = I$ from the hypothesis we have $(z \frown y) \smile (y \frown a) = z \smile (y \frown a) \leqq (z \smile y) \frown (z \smile a) = z \smile y = y$, and hence from $y \frown a \in B(a, b)$ we have $z \smile (y \frown a) = y$ since $x \smile z = I$ or $x \smile z = y$ for $x \in B(a, b)$. Thus we have zya. We have zyb from $z \smile b = y$. Accordingly by Lemma 11 we have zyc for $c \in B(a, b)$, that is, y is not an *extreme element*.

Theorem 9. *A compact lattice which is an extreme lattice is a complemented lattice.*

Proof. Let L be an *extreme lattice*; then if we take $c \neq O, I$ of L, then there exists $x_1 \in L$ such that $(x_1, c)E$.

Case I. If $B^*(c, x_1) = x_1$ and if $c \smile x_1 = I$, $c \frown x_1 = O$, then x_1 is the complement of c. If $c \smile x_1 < I$, $c \frown x_1 > O$, then let $c \smile x_1 = a$, $c \frown x_1 = b$. In this case there exists z such that $z \smile a = I$ and $z \frown a = O$, for otherwise L is not an *extreme lattice* by Lemma 12. Hence we have $z \smile c = I$, $z \frown c = O$ by Lemma 12.

Case II. $B^*(c, x_1) \ni x_2, \cdots, B^*(c, x_{n-1}) \ni x_n, \cdots$. Since L is a compact lattice $c \frown x_n \to c \frown x_0$ and $c \smile x_n \to c \smile x_0$ as x_n tends to x_0. From $cx_1x_2, cx_2x_3, \cdots, cx_{n-1}x_n, \cdots$, we have

$$c \smile x_1 \leqq c \smile x_2 \leqq \cdots \leqq c \smile x_n \leqq \cdots \leqq c \smile x_0;$$

$$c \frown x_1 \geqq c \frown x_2 \geqq \cdots \geqq c \frown x_n \geqq \cdots \geqq c \frown x_0 \text{ by Lemma 3.}$$

Thus we have $(c, x_0)E$. Then if $c \smile x_0 < I$, $c \frown x_0 > O$, we can find the complement of c in the same way as in Case I.

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