44. *Notes on Uniform Convergence of Trigonometrical Series. I*

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1. In the preceding paper [1] we have studied the uniform convergence of the series

$$\sum_{n=1}^{\infty} \frac{s_n}{n} \sin nt$$

concerning the Riemann summability (R_1). In this paper we shall treat the cosine-analogue.

Let $\{s_n;\ n=1, 2, \cdots\}$ be a sequence with real terms, and let

$$s_n^{\gamma}=\sum_{\nu=0}^{n} A_{n-\nu}^{\gamma-1}\, s_\nu \qquad (-\infty<\gamma<\infty),$$

where $s_0=0$ and $A_n^{\gamma}=\binom{\gamma+n}{n}$. The theorem to be proved is as follows:

THEOREM 1. Suppose that $0<r$, $0<s<1$ (or $s=1, 2, \cdots$), and $0<\alpha\leqq 1$, and that

$$\sum_{\nu=1}^{n} |s_\nu^r| = o(n^{1+r\alpha}), \tag{1.1}$$

$$\sum_{\nu=n}^{2n} (|s_\nu^{-s}|-s_\nu^{-s})=O(n^{1-s\alpha}), \tag{1.2}$$

as $n\to\infty$. Then, (I) when $0<\alpha<1$ the series

$$\sum_{n=1}^{\infty} \frac{s_n}{n} \cos nt \tag{1.3}$$

converges uniformly (on the real axis), and (II) when $\alpha=1$ the series (1.3) converges uniformly if and only if $\sum n^{-1}s_n$ converges.

COROLLARY 1. If

$$\sum_{\nu=n}^{2n} (|s_\nu^{-1}|-s_\nu^{-1})=O(1) \qquad (n\to\infty),$$

where $s_n^{-1}=s_n-s_{n-1}$, and if the series in

$$g(t)=\sum_{n=1}^{\infty} s_n \sin nt \tag{1.4}$$

converges boundedly in the interval (δ, π) for any $\delta>0$, then a necessary and sufficient condition for the convergence of the Caucy integral

$$\int_{\to 0}^{\pi} g(t)\, dt \tag{1.5}$$

is the convergence of the series $\sum n^{-1}s_n$.

This is a theorem of Izumi [2, 3].

This corollary follows from Theorem 1 with $r=s=\alpha=1$, since the convergence of the series in (1.4) implies $s_n=o(1)$, cf. Zygmund [4,

p. 268], and of course $s_n^1=o(n)$, and the convergence of (1.5) is equivalent to the existence of

$$\lim_{t\to+0}\sum_{n=1}^{\infty}\frac{s_n}{n}\cos nt$$

by Lemma 1 below.

2. In order to prove Theorem 1 we need some lemmas.

LEMMA 1. Suppose that $0<r$, $0<s$ and $0<\alpha\leqq 1$. Then the two conditions (1.1) and (1.2) imply the convergence of the series (1.3) in (δ,π) for any $\delta>0$, and in particular the convergence of

$$\sum_{n=1}^{\infty}(-1)^n\frac{s_n}{n}. \tag{2.1}$$

LEMMA 1.1. If $\sum c_n(1-\cos nx)$ is convergent for all x of an interval (α,β), then $\sum c_n$ is convergent.

This is Theorem 258 in Hardy [5, p. 366].

Proof of Lemma 1. Observe that by Abel's transformation

$$\begin{aligned}\sum_{\nu=1}^{n}s_\nu e^{i\nu u}=&(1-e^{iu})^{-s}\sum_{\mu=1}^{n}s_\mu^{-s}e^{i\mu u}\\&-(1-e^{iu})^{-[s]}\sum_{\mu=1}^{n}s_\mu^{-s}\sum_{\nu=n+1}^{\infty}A_{\nu-\mu}^{s-[s]-1}e^{i\nu u}\\&-\sum_{j=1}^{[s]}s_n^{1-j}(1-e^{iu})^{-j}e^{i(n+1)u},\end{aligned}$$

where the second term vanishes when s is integral, and the third term does when $0<s<1$. Using this identity and repeating the argument in Yano [1], we see that under the assumption in the lemma

$$\sum_{\nu=1}^{\infty}s_\nu\int_t^{\pi}e^{i\nu u}\,du \tag{2.2}$$

converges in the interval $\delta\leqq t\leqq\pi$ for every δ such as $0<\delta<\pi$. Here we do not reproduce the argument. The convergence of (2.2) in (δ,π) implies that of the series

$$\begin{aligned}\sum_{\nu=1}^{\infty}s_\nu\int_t^{\pi}\sin\nu u\,du&=-\sum_{\nu=1}^{\infty}(-1)^\nu\frac{s_\nu}{\nu}[1-\cos\nu(t+\pi)]\\&=-\sum_{\nu=1}^{\infty}(-1)^\nu\frac{s_\nu}{\nu}(1-\cos\nu x),\ x=t+\pi,\end{aligned}$$

in $\delta+\pi\leqq x\leqq 2\pi$. From this follows the convergence of the series (2.1) by Lemma 1.1, and we get the desired result.

LEMMA 2. Suppose that $0<r$, $0<s$ and $0<\alpha<1$, then the two conditions (1.1) and (1.2) imply the convergence of the series

$$\sum_{n=1}^{\infty}\frac{s_n}{n}. \tag{2.3}$$

In the case $\alpha=1$, this lemma is not true. This is easily seen by taking the sequence $s_0=s_1=0$ and $s_n=1/\log n$ for $n\geqq 2$.

LEMMA 2.1. Suppose that $0<r$, $0<s$ and $0<\alpha\leqq 1$. Then the two conditions (1.1) and (1.2) imply

(2.4) $$s_n^{1+\mu} = o(n^{1+\mu\alpha}) \qquad -s<\mu\leqq r.$$

Concerning this lemma, cf. Lemma 2 in Yano [1].

Proof of Lemma 2. If $r\geqq 1$ we have $s_n^2=o(n^{1+\alpha})$ from (2.4) with $\mu=1$. And if $r<1$ (2.4) with $\mu=r$ yields $s_n^{1+r}=o(n^{1+r\alpha})$, and then

$$s_n^2=s_n^{1+r+(1-r)}=o(n^{1+r\alpha+(1-r)}).$$

Hence, in both cases we get for some δ such as $0<\delta<1$,

(2.5) $$s_n^2 = o(n^{1+\delta}).$$

From (2.5) and $s_n^1=o(n)$ which is (2.2) with $\mu=0$, it follows

$$\sum_{\nu=n+1}^{n+m}\frac{s_\nu}{\nu}=2\sum_{\nu=n+1}^{n+m}\frac{s_\nu^2}{\nu(\nu+1)(\nu+2)}+\frac{s_{n+m}^2}{(n+m+1)(n+m+2)}$$
$$-\frac{s_n^2}{(n+1)(n+2)}+\frac{s_{n+m}^1}{n+m+1}-\frac{s_n^1}{n+1}=o(1)$$

as $n\to\infty$ for $m=1,2,\cdots$. This proves the convergence of the series (2.3), and we get the lemma.

Proof of Theorem 1. The proof runs analogously as Theorem 1 in Yano [1], whose proof is essentially based on the estimation of the two expressions

$$\sum_{\nu=1}^{n}s_\nu\int_0^t e^{i\nu u}\,du \quad\text{and}\quad \sum_{\nu=n+1}^{\infty}s_\nu\int_t^\pi e^{i\nu u}\,du.$$

Now, if the series (1.3), i.e.,

(2.6) $$\sum_{\nu=1}^{\infty}\frac{s_\nu}{\nu}\cos\nu t$$

converges uniformly (on the real axis), then the series $\sum\nu^{-1}s_\nu$ necessarily converges.

Inversely, if $\sum\nu^{-1}s_\nu$ converges the uniform convergence of (2.6) is equivalent to that of the series in

(2.7) $$\sum_{\nu=1}^{\infty}\frac{s_\nu}{\nu}(1-\cos\nu t)=\sum_{\nu=1}^{n}+\sum_{\nu=n+1}^{\infty}=S_n+R_n,$$

where

$$S_n=\Im\Big(\sum_{\nu=1}^{n}s_\nu\int_0^t e^{i\nu u}\,du\Big),$$

$$R_n=\Im\Big(-\sum_{\nu=n+1}^{\infty}s_\nu\int_t^\pi e^{i\nu u}\,du\Big)+\sum_{n+1}^{\infty}(-1)^\nu\frac{s_\nu}{\nu}-\sum_{n+1}^{\infty}\frac{s_\nu}{\nu}.$$

And, under the conditions in the theorem, the series $\sum(-1)^\nu\nu^{-1}s_\nu$ converges by Lemma 1, and $\sum\nu^{-1}s_\nu$ does by the above assumption. Hence, the uniform convergence of the series in (2.7) is certainly verified by the argument used in loc. cit. [1].

Combining the above result with Lemma 2 we get the theorem.

3. In the case $r=0$ Theorem 1 is not true, and we can then prove the following theorem quite similarly. Cf. also Theorem 3 in Yano [1].

THEOREM 2. If

$$\sum_{\nu=n}^{2n} |s_\nu| = o(n/\log n) \qquad (n \to \infty),$$

and if for some positive s and δ

$$\sum_{\nu=n}^{2n} (|s_\nu^{-s}| - s_\nu^{-s}) = O(n^{1-\delta}) \qquad (n \to \infty),$$

then the series $\sum n^{-1}s_n \cos nt$ converges uniformly if and only if $\sum n^{-1}s_n$ converges.

Letting $s=1$ and $n^{-1}s_n = a_n$ in Theorem 3 in Yano [1] and the above Theorem 2, and after some modification we have the following

COROLLARY 2. Suppose that

$$\sum_{\nu=n}^{2n} |a_\nu| = o(1/\log n),$$

and that for some positive δ

$$\sum_{\nu=n}^{2n} (|\Delta a_\nu| - \Delta a_\nu) = O(n^{-\delta}),$$

where $\Delta a_n = a_n - a_{n+1}$. Then, (I) the sine series $\sum a_n \sin nt$ converges uniformly, and (II) the cosine series $\sum a_n \cos nt$ converges uniformly if and only if $\sum a_n$ converges.

References

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