57. Notes on Uniform Convergence of Trigonometrical Series. II

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1. We consider a series with real terms

$$\sum_{n=1}^{\infty} a_n \qquad (a_0=0),$$

and write

$$(1.1) \qquad s_n^{\gamma}=\sum_{\nu=0}^{n} A_{n-\nu}^{\gamma}\, a_\nu=\sum_{\nu=0}^{n} A_{n-\nu}^{\gamma-1}\, s_\nu$$

$$(-\infty<\gamma<\infty),$$

$$(1.2) \qquad t_n^{\gamma}=\sum_{\nu=0}^{n} A_{n-\nu}^{\gamma-1}(\nu a_\nu)=\sum_{\nu=0}^{n} A_{n-\nu}^{\gamma-1}\, t_\nu$$

where $s_n=s_n^0$, $t_n=t_n^0$, and $A_n^{\gamma}=\binom{\gamma+n}{n}$. Then, in particular $s_0^{\gamma}=0$, $t_0^{\gamma}=0$, and for $n=1,2,\cdots$,

$$s_n^{-1}=a_n, \quad s_n^{-2}=a_n-a_{n-1}=-\Delta a_{n-1},$$
$$t_n^0=na_n, \quad t_n^{-1}=na_n-(n-1)a_{n-1}.$$

The object of this paper is to prove some theorems (Theorems 3-5) which will unify the results of Szász [1], Hirokawa [5] and others. This note is a continuation of Yano [6, 7].

THEOREM 1. Let $0<r$, $0<s<1$ (or $s=1,2,\cdots$) and $0<\alpha\leqq 1$. If

$$(1.3) \qquad \sum_{\nu=1}^{n} |t_\nu^r|=o(n^{1+r\alpha}),$$

$$(1.4) \qquad \sum_{\nu=n}^{2n} (|t_\nu^{-s}|-t_\nu^{-s})=O(n^{1-s\alpha}),$$

as $n\to\infty$, then the series $\sum a_n \sin nt$ converges uniformly (on the real axis).

THEOREM 2. Under the same assumption as in Theorem 1, the series $\sum a_n \cos nt$ converges uniformly when $0<\alpha<1$, and in the case $\alpha=1$ this series converges uniformly if and only if $\sum a_n$ converges.

These theorems are an alternative form of Theorem 1 in the papers [6] and [7] respectively.

2. THEOREM 3. Let $0<s\leqq 1$, and q be an arbitrary real constant. If

$$(\mathrm{A}.2) \qquad (1-x)\sum_{n=1}^{\infty} na_n x^n \to 0 \qquad (x\to 1-0),$$

$$(2.1) \qquad \sum_{\nu=n}^{2n} (|\gamma_\nu|-\gamma_\nu)=O(n^{1-s}) \qquad (n\to\infty),$$

where

$$(2.2) \qquad \gamma_n=(1+qn^{-1})t_n^{1-s}-t_{n+1}^{1-s} \qquad (n=1,2,\cdots),$$

then, (I) $\sum a_n \sin nt$ converges uniformly, and (II) $\sum a_n \cos nt$ converges uniformly if and only if $\sum a_n$ converges.

COROLLARY 3.1. Let p and q be two arbitrary real constants, then the condition (A.2), and

$$\sum_{\nu=n}^{2n}(|\gamma_\nu|-\gamma_\nu)=O(1), \quad \text{where} \tag{2.3}$$

$$\gamma_n=(1+qn^{-1})(na_n+p)-[(n+1)a_{n+1}+p], \tag{2.4}$$

imply the conclusion of Theorem 3.

This is a result from Theorem 3 with $s=1$, and this corollary contains a theorem of Szász [1], in which the condition (2.3) with (2.4) is replaced by "$p\geqq 0$, $q\geqq 0$, and for $n\geqq n_0$

$$0\leqq(n+1)a_{n+1}+p\leqq(1+qn^{-1})(na_n+p)\text{''}.$$

COROLLARY 3.2. The condition (A.2) and

$$\sum_{\nu=n}^{2n}(|\Delta a_\nu|-\Delta a_\nu)=O(n^{-1}) \qquad (n\to\infty), \tag{2.5}$$

imply the uniform convergence of $\sum a_n \sin nt$.

This follows from Corollary 3.1 with $p=0$ and $q=1$, since then $\gamma_n=(n+1)\Delta a_n$.

Proof of Theorem 3. The theorem follows immediately from Theorems 1, 2 with $\alpha=1$, and the following lemma.

LEMMA 1. The assumption in Theorem 3 implies $t_n^1=o(n)$, and

$$\sum_{\nu=n}^{2n}(|t_\nu^{-s}|+t_\nu^{-s})=O(n^{1-s}). \tag{2.6}$$

For the proof of this lemma we need some other lemmas.

LEMMA 1.1. If $\alpha>0$, and s_n^α is defined by (1.1), then the Abel summability of $\sum a_n$, i.e.

$$(1-x)\sum_{n=1}^{\infty}s_n x^n\to C \qquad (x\to 1-0) \tag{A.1}$$

implies
$$(1-x)\sum_{n=1}^{\infty}(s_n^\alpha/A_n^\alpha)x^n\to C \qquad (x\to 1-0).$$

This is due to Szász [3].

LEMMA 1.2. If (A.1) holds, and $s_n=O_L(1)$, then $s_n^1\sim Cn$ as $n\to\infty$.

This appears in Hardy [9, p. 155].

LEMMA 1.3. If $u_\nu\geqq 0$ and $\alpha>0$, then

$$\sum_{\nu=n}^{2n}u_\nu=O(n^\alpha)\Longleftrightarrow\sum_{\nu=1}^{n}u_\nu=O(n^\alpha),$$

$$\sum_{\nu=n}^{2n}u_\nu=O(n^{-\alpha})\Longleftrightarrow\sum_{\nu=n}^{\infty}u_\nu=O(n^{-\alpha}),$$

as $n\to\infty$. O's may be replaced by o's respectively.

This is Lemma 1 in Yano [6].

Proof of Lemma 1. γ_n in (2.2) is written as

$$\gamma_n=(\Gamma(n+1+q)/\Gamma(n+1))\Delta c_n, \tag{2.7}$$

where $\Delta c_n=c_n-c_{n+1}$, and

$$c_n=(\Gamma(n)/\Gamma(n+q))t_n^{1-s}. \tag{2.8}$$

Here we may suppose that $c_0=0$ when $q>-1$, and $c_0, c_1, \cdots, c_{[-q]}$ are all zero when $q\leqq -1$. This assumption is permissible with no loss of generality as the succeeding argument shows. Observing that $\Gamma(n+q)/\Gamma(n)\sim n^q$ by Stirling's formula, the condition (2.1) is, by (2.7), equivalent to

$$\sum_{\nu=n}^{2n}(|\Delta c_\nu|-\Delta c_\nu)=O(n^{1-s-q}). \tag{2.9}$$

Now, the condition (A.2), i.e. $(1-x)\sum t_n x^n\to 0$ implies

$$(1-x)\sum_{n=1}^{\infty}(t_n^{1-s}/A_n^{1-s})x^n\to 0 \qquad (x\to 1-0), \tag{2.10}$$

by Lemma 1.1, since $1-s\geqq 0$, and (2.10) is written as

$$(1-x)\sum_{n=1}^{\infty}(\Gamma(n+q)/\Gamma(n)A_n^{1-s})c_n x^n\to 0 \tag{2.11$'$}$$

by (2.8). Further, observing that $\Gamma(n+q)/\Gamma(n)A_n^{1-s}\sim\Gamma(2-s)n^{s+q-1}$, we may for the sake of convenience replace (2.11)$'$ by

$$(1-x)\sum_{n=1}^{\infty}n^{s+q-1}c_n x^n\to 0 \qquad (x\to 1-0). \tag{2.11}$$

If $1-s-q<0$, applying Lemma 1.3 to (2.9) we have

$$\sum_{\nu=n}^{n+m-1}|\Delta c_\nu|-(c_n-c_{n+m})<Cn^{1-s-q},\ C>0, \tag{2.12}$$

for all $m>0$, and then successively

$$c_n-c_{n+m}>-Cn^{1-s-q} \qquad (m=1, 2, \cdots),$$
$$c_n\geqq \limsup c_n-Cn^{1-s-q},$$
$$\liminf c_n\geqq \limsup c_n.$$

This implies the existence of $\lim c_n$ which may be finite or $-\infty$, and this limit must vanish by (2.11), since if otherwise we have a contradiction. So, letting $m\to\infty$, (2.12) yields

$$\sum_{\nu=n}^{\infty}|\Delta c_\nu|-c_n\leqq Cn^{1-s-q}.$$

Combining this inequality with (2.11) we get

$$(1-x)\sum_{n=1}^{\infty}n^{s+q-1}\Big(\sum_{\nu=n}^{\infty}|\Delta c_\nu|-Cn^{1-s-q}\Big)x^n$$
$$\leqq (1-x)\sum_{n=1}^{\infty}n^{s+q-1}c_n x^n\to 0 \qquad (x\to 1-0),$$

i.e.
$$(1-x)\sum_{n=1}^{\infty}\Big(n^{s+q-1}\sum_{\nu=n}^{\infty}|\Delta c_\nu|\Big)x^n<C \qquad (0\leqq x<1),$$

where and in the sequel the constant C may be different in different cases. Since the coefficients of x^n are all positive we get by an analogue to Lemma 1.2,

$$\sum_{\mu=1}^{n}\Big(\mu^{s+q-1}\sum_{\nu=\mu}^{\infty}|\Delta c_\nu|\Big)<Cn.$$

From this inequality replaced the lower limit $\nu=\mu$ in the second sum $\sum_{\nu=\mu}^{\infty}$ by $\nu=n$ it follows

$$n^{s+q}\sum_{\nu=n}^{\infty}|\Delta c_\nu|<Cn,$$

which and $c_n \to 0$ imply $c_n = O(n^{1-s-q})$.

(2.8) and $c_n = O(n^{1-s-q})$ yield

(2.13) $$t_n^{1-s} = O(n^{1-s}), \text{ i.e. } t_n^{1-s}/A_n^{1-s} = O(1).$$

Applying Lemma 1.2 to (2.10) and (2.13) we have $\sum_{\nu=1}^{n}(t_\nu^{1-s}/A_\nu^{1-s}) = o(n)$, which is equivalent to $t_n^{2-s} = o(A_n^{2-s})$ by the well-known property between Cesàro's summation and Hölder's. $t_n^{2-s} = o(n^{2-s})$ and (2.13) imply $t_n^{1-s+\delta} = o(n^{1-s+\delta})$ for every $\delta > 0$ by a convexity theorem of Tauberian type, and in particular

(2.14) $$t_n^1 = o(n).$$

Further, γ_{n-1} in (2.2) is

$$\gamma_{n-1} = -t_n^{-s} + qn^{-1}t_{n-1}^{1-s} = -t_n^{-s} + O(n^{-s})$$

by (2.13). Hence, the proposition (2.6) follows from the last relation and (2.1), since

$$\sum_{\nu=n}^{2n}(|t_\nu^{-s}| + t_\nu^{-s}) = \sum_{\nu=n}^{2n}[|\gamma_{\nu-1} + O(\nu^{-s})| - \gamma_{\nu-1} - O(\nu^{-s})]$$
$$\leqq \sum_{\nu=n}^{2n}[(|\gamma_{\nu-1}| - \gamma_{\nu-1}) + O(\nu^{-s})] = O(n^{1-s}).$$

This and (2.14) prove the lemma in the present case.

If $1-s-q>0$, applying Lemma 1.3 to (2.9) we have $\sum_{\nu=0}^{n-1}|\Delta c_\nu| + c_n < Cn^{1-s-q}$. Substituting this inequality into (2.11),

$$(1-x)\sum_{n=1}^{\infty}\left(n^{s+q-1}\sum_{\nu=0}^{n-1}|\Delta c_\nu|\right)x^n < C.$$

Thus, $$\sum_{\mu=1}^{2n}\left(\mu^{s+q-1}\sum_{\nu=0}^{\mu-1}|\Delta c_\nu|\right) < Cn,$$

again by an analogue to Lemma 1.2, and so replacing the lower limit $\mu=1$ in $\sum_{\mu=1}^{2n}$ by $\mu=n$,

$$n^{s+q}\sum_{\nu=0}^{n-1}|\Delta c_\nu| < Cn.$$

This implies $c_n = O(n^{1-s-q})$, and the conclusion is the same as the case $1-s-q<0$.

Finally, if $1-s-q=0$ then (2.11) and (2.9) are reduced to

$$(1-x)\sum_{n=1}^{\infty}c_n x^n \to 0 \qquad (x \to 1-0),$$

and $$\sum_{\nu=n}^{2n}(|\Delta c_\nu| - \Delta c_\nu) = O(1) \qquad (n \to \infty),$$

respectively. These two conditions imply $c_n = O(1) = O(n^{1-s-q})$, by a lemma (Lemma 1) due to Szász [2]. Hence, in this case also the conclusion is the same as the case $1-s-q<0$. Thus the lemma is established completely.

3. Using Theorems 1, 2 and the preceding lemmas we can prove the following theorem analogously as Theorem 3.

THEOREM 4. Let $0 < s \leqq 1$, and p, q be two arbitrary constants. If

(A.1) $$(1-x)\sum_{n=1}^{\infty}s_n x^n \to \sigma \qquad (x \to 1-0),$$

and
$$\sum_{\nu=n}^{2n}(|\delta_\nu|-\delta_\nu)=O(n^{1-s}) \qquad (n\to\infty),$$
where
$$(3.1)\qquad \delta_n=(1+qn^{-1})(t_n^{1-s}+ps_{n-1}^{1-s})-(t_{n+1}^{1-s}+ps_n^{1-s}),$$
then $s_n\to\sigma$, and the series $\sum a_n e^{int}$ converges uniformly (on the real axis).

COROLLARY 4.1. Let p and q be two arbitrary constants, then the condition (A.1) and
$$(3.2)\qquad \sum_{\nu=n}^{2n}(|\delta_\nu|-\delta_\nu)=O(1), \quad \text{where}$$
$$(3.3)\qquad \delta_n=(1+qn^{-1})[ns_n-(n-1)s_{n-1}+p]-[(n+1)s_{n+1}-ns_n+p],$$
imply $s_n\to\sigma$, and the uniform convergence of $\sum a_n e^{int}$

This follows from Theorem 4 with $s=p=1$, and contains a theorem of Szász [1], in which the condition (3.2) with (3.3) is replaced by "$p\geqq 0$, $q\geqq 0$, and for $n\geqq n_0$
$$0\leqq(n+1)s_{n+1}-ns_n+p\leqq(1+qn^{-1})[ns_n-(n-1)s_{n-1}+p]\text{''}.$$

COROLLARY 4.2. The condition (A.1) and
$$(3.4)\qquad \sum_{\nu=n}^{2n}(|s_\nu^{-s-1}|+s_\nu^{-s-1})=O(n^{-s}), \qquad 0<s\leqq 1,$$
imply the uniform convergence of $\sum a_n e^{int}$

This follows from Theorem 4 with $p=1-s$ and $q=1$, since then the identity $t_n^r=ns_n^{r-1}-\gamma s_{n-1}^r$ implies $\delta_{n-1}=-ns_n^{-s-1}$ The case $s=1$ is as follows:

COROLLARY 4.3. The condition (A.1) and
$$\sum_{\nu=n}^{2n}(|\Delta a_\nu|-\Delta a_\nu)=O(n^{-1})$$
imply the uniform convergence of $\sum a_n e^{int}$

Remark. We see from Corollary 4.2 that "if $\sum a_n$ is summable $(C, -1-\delta)$ for some positive δ, then the series $\sum a_n \cos nt$ and $\sum a_n \sin nt$ converge uniformly" as it is known. But this is not true when $\delta=0$, since then a negative example has been given by Izumi [4] for the cosine series, and by Hardy-Littlewood [8] for the sine series.

Theorems 3, 4 are concerned with the case $\alpha=1$ in Theorems 1, 2. In the case $0<\alpha<1$ we have the following

THEOREM 5. Let $0<r$, $0<s<1$ (or $s=1,2,\cdots$), and $0<\alpha<1$. If
$$\sum_{\nu=1}^{n}|t_\nu^r|=o(n^{1+r\alpha}),$$
and
$$\sum_{\nu=n}^{2n}(|\delta_\nu|-\delta_\nu)=O(n^{1-s\alpha}),$$
where δ_n is defined by (3.1), then $\sum a_n$ converges, and the series $\sum a_n e^{int}$ converges uniformly.

COROLLARY 5. If $0<r$, $0<\alpha<1$, and $t_n^r=o(n^{r\alpha})$ and
$$(3.5)\qquad \sum_{\nu=n}^{2n}(|\Delta a_\nu|-\Delta a_\nu)=O(n^{-\alpha}),$$

then $\sum a_n e^{int}$ converges uniformly.

This corollary is due to Hirokawa [5] when (3.5) is replaced by $\sum_{\nu=n}^{2n}|\Delta a_\nu|=O(n^{-\alpha})$.

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