81. On the Singularity of a Positive Linear Functional on Operator Algebra

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In the previous paper [3], we have introduced the notion of a singular linear functional on W^*-algebra $\boldsymbol{M}$ as follows: a positive linear functional φ on $\boldsymbol{M}$ is called *singular* if there exists no non-zero σ-weakly continuous positive linear functional ψ such as $\psi\leqq\varphi$. This notion is corresponding to the one of *purely finite additive measure* in the abelian case of Yosida-Hewitt [5]. And we have proved the decomposition theorem of positive linear functional on $\boldsymbol{M}$, whose another proof was given by Nakamura in [2], as follows: *Any positive linear functional φ on $\boldsymbol{M}$ is uniquely decomposed into the sum of σ-weakly continuous positive linear functional φ_1 and singular one φ_2. And if φ is singular, then φ is so on $p\boldsymbol{M}p$ for every non-zero projection p of $\boldsymbol{M}$.* Moreover, suppose $\boldsymbol{M}_*$ is the space of all σ-weakly continuous linear functionals on $\boldsymbol{M}$ and $\boldsymbol{M}_*^\perp$ the space of all linear combinations of singular positive linear functionals, we have proved the following decomposition of the conjugate space $\boldsymbol{M}^*$ of $\boldsymbol{M}$: $\boldsymbol{M}^*=\boldsymbol{M}_*\oplus_{l^1}\boldsymbol{M}_*^\perp$ *where $\oplus_{l^1}$ means the l^1-direct sum of its summands.* This decomposition of the conjugate space implies that of a uniformly continuous mapping which proved by Tomiyama [4] as follows: *Let π be a uniformly continuous linear mapping from $\boldsymbol{M}$ into another W^*-algebra $\boldsymbol{N}$, then there exist unique two linear mappings π_1 and π_2 of $\boldsymbol{M}$ into $\boldsymbol{N}$ such that $\pi=\pi_1+\pi_2$, π_1 is σ-weakly continuous and ${}^t\pi_2(\boldsymbol{N}_*)\subset\boldsymbol{M}_*^\perp$ where ${}^t\pi_2$ means the transpose of π_2. And according to π being a homomorphism, positive or *-preserving, π_1 and π_2 are homomorphisms, positive or *-preserving respectively.* Hence a linear functional φ on M and a linear mapping π from $\boldsymbol{M}$ into another W^*-algebra $\boldsymbol{N}$ are called *singular* if $\varphi\in\boldsymbol{M}_*^\perp$ and ${}^t\pi(\boldsymbol{N}_*)\subset\boldsymbol{M}_*^\perp$, respectively.

This note is devoted to give a characterization of the singularity of a positive linear functional on $\boldsymbol{M}$ and a short alternative proof of Theorem 6 in [3].

Theorem 1. *Let $\boldsymbol{M}$ be a W^*-algebra and φ a positive linear functional on $\boldsymbol{M}$. Then a necessary and sufficient condition that φ is singular is that for any non-zero projection e, there exists a non-zero projection $f\leqq e$ such as $\langle f,\varphi\rangle=0$.*

Proof. Suppose φ is not singular, the σ-weakly continuous part φ_1 of φ is not zero by Theorem 3 in [3]. Let e be the carrier projection of φ_1, then φ_1 is faithful on $e\boldsymbol{M}e$. Hence the inequality $\varphi \geqq \varphi_1$ implies the faithfulness of φ on $e\boldsymbol{M}e$.

Next suppose φ is singular and e any fixed projection of $\boldsymbol{M}$. Take a σ-weakly continuous positive linear functional ψ such as $\langle e, \varphi\rangle < \langle e, \psi\rangle$. The family $\mathcal{F}$ of all projections p of $\boldsymbol{M}$ such as $p \leqq e$ and $\langle p, \varphi\rangle \geqq \langle p, \psi\rangle$ is an inductive set relative to the natural ordering. In fact, if $\{p_\alpha\}$ is a totally ordered subfamily of $\mathcal{F}$, we put $p = \sup_\alpha p_\alpha$. Then $\langle p, \varphi\rangle \geqq \sup_\alpha \langle p_\alpha, \varphi\rangle \geqq \sup_\alpha \langle p_\alpha, \psi\rangle = \langle p, \psi\rangle$, so that p belongs to $\mathcal{F}$. Therefore there exists a maximal projection p_0 of $\mathcal{F}$ by Zorn's lemma. Putting $f = e - p_0$, we have $f \neq 0$ by the hypothesis for ψ. Moreover the maximality of p_0 implies $\langle q, \varphi\rangle < \langle q, \psi\rangle$ for every non-zero projection $q \leqq f$, so that we have $\varphi < \psi$ on $f\boldsymbol{M}f$ by the usual spectral theorem. Therefore φ is σ-weakly continuous on $f\boldsymbol{M}f$, so that φ vanishes on $f\boldsymbol{M}f$ by the singularity of φ. This concludes the proof.

This theorem is clearly equivalent to Theorem 4 of [3] in the abelian case. Moreover, using this characterization of the singularity, we can directly conclude Theorem 6 of [3] without using a maximal abelian subalgebra. That is,

Theorem 2. *Let $\boldsymbol{M}$ and $\boldsymbol{N}$ be two σ-finite W^*-algebras and π a faithful positive linear mapping from $\boldsymbol{M}$ into $\boldsymbol{N}$, then there exists a non-zero projection e of $\boldsymbol{M}$ such that π is σ-weakly continuous on $e\boldsymbol{M}e$. And the σ-weakly continuous part π_1 of π is faithful on $\boldsymbol{M}$ and the singular part π_2 of π not faithful on $\boldsymbol{M}$.*

Proof. There exists a faithful σ-weakly continuous positive linear functional ψ on $\boldsymbol{N}$ by the σ-finiteness of $\boldsymbol{N}$. Putting $\varphi = {}^t\pi(\psi)$, $\varphi_1 = {}^t\pi_1(\psi)$ and $\varphi_2 = {}^t\pi_2(\psi)$, φ_1 is the σ-weakly continuous part of φ and φ_2 the singular part of φ respectively. Then we have

$$\langle x^*x, \varphi\rangle = \langle \pi(x^*x), \psi\rangle \neq 0$$

for every non-zero $x \in \boldsymbol{M}$ by the faithfulness of π and ψ, so that φ is faithful on $\boldsymbol{M}$. Hence φ_1 is faithful and there exists a projection e of $\boldsymbol{M}$ such that $\langle e, \varphi_2\rangle = 0$ by Theorem 1, which imply that π_1 is faithful and $\pi_2(e) = 0$. Therefore π coincides with π_1 on $e\boldsymbol{M}e$, so that π is σ-weakly continuous on $e\boldsymbol{M}e$. This concludes the proof.

References

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