76. On Quasi-normed Space. I

By Tomoko KONDA

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Recently a linear metric space which is defined by a quasi-norm was considered by M. Pavel [1] and S. Rolewicz [2]. In this paper, we shall consider such a new linear space with the metric, and establish some results.

Definition 1. *Let E be a linear space over the real field Φ. A real function $\|x\|$ of x is called a quasi-norm with the power r if it satisfies the following conditions.*

1° $\|x+y\|\leq\|x\|+\|y\|$, *for any* $x, y\in E$.

2° $\|\lambda x\|=|\lambda|^r\|x\|$, *for* $\lambda\in\Phi$ *and* $x\in E$, $0<r\leq 1$.

3° $\|x\|=0$ *if and only if* $x=0$.

Let $\|x\|$ be a quasi-norm with the power r and let $d(x,y)=\|x-y\|$, $x\in E$, $y\in E$, then d is distance in E. A linear topological space which is defined by the distance d is called a quasi-normed space with the power r.

Definition 2. *Let E be a quasi-normed space with the power r and if E is complete with the distance d. E will be called a (QN) space with the power r.*

a) By the trivial relations:

$$\|(x+y)-(x_n+y_n)\|\leq\|x-x_n\|+\|y-y_n\|$$

$$\|\lambda x-\lambda_n x_n\|\leq\|\lambda x-\lambda_n x\|+\|\lambda_n x-\lambda_n x_n\|\leq|\lambda-\lambda_n|^r\|x\|+|\lambda_n|^r\|x-x_n\|,$$

if $\lambda_n\to\lambda$, $x_n\to x$ and $y_n\to y$, then we have the convergence $x_n+y_n\to x+y$, $\lambda_n x_n\to\lambda x$. Hence $(x,y)\to x+y$. $(\lambda,x)\to\lambda x$ are continuous on two variables.

b) From the relation:

$$\|x\|=\|y+(x-y)\|\leq\|y\|+\|x-y\|$$

we have

$$\|x\|-\|y\|\leq\|x-y\|.$$

If we replace x and y, then we have

$$\|y\|-\|x\|\leq\|x-y\|.$$

Thus we have the following relation $|\,\|x\|-\|y\|\,|\leq\|x-y\|$.

c) If $x_n\to x$, then $\|x_n\|\to\|x\|$.

This follows from b).

Let E be a topological space, R an equivalent relation on E. We generate the quotient topology on the quotient set E/R. It is the strongest topology in which canonical map φ of E on E/R is continuous. The set E/R, with this topology, is called the quotient space of E by R. If E is a quasi-normed space, then the norm of a coset $\dot{x}$ in a quotient space E/R is defined as

$$\|\dot{x}\|=\inf\{\|x\|;\ x\in\dot{x}\}.$$

Then we have the following theorem.

Theorem. *If E is a (QN) space with the power r and N a closed subspace, then the quotient space E/N is a (QN) space with the power r.*

Proof. First, $\|\dot{x}\|=0$ if and only if there exists a sequence $x_n\in\dot{x}$ such that $\|x_n\|\to 0$. Since $\dot{x}$ is closed, there exists a limit 0, so that $\|\dot{x}\|=0$ if and only if $\dot{x}=N$.

Next,

$$\begin{aligned}\|\dot{x}_1+\dot{x}_2\|&=\inf\{\|x_1+x_2\|;\ x_1\in\dot{x}_1,\ x_2\in\dot{x}_2\}\\&\leq\inf\{\|x_1\|+\|x_2\|;\ x_1\in\dot{x}_1,\ x_2\in\dot{x}_2\}\\&=\inf\{\|x_1\|;\ x_1\in\dot{x}_1\}+\inf\{\|x_2\|,\ x_2\in\dot{x}_2\}\\&=\|\dot{x}_1\|+\|\dot{x}_2\|.\end{aligned}$$

Similarly,

$$\begin{aligned}\|\lambda\dot{x}\|&=\inf\{\|\lambda x\|;\ x\in\dot{x}\}=\inf\{|\lambda|^r\|x\|;\ x\in\dot{x}\}\\&=|\lambda|^r\inf\{\|x\|;\ x\in\dot{x}\}=|\lambda|^r\|\dot{x}\|\end{aligned}$$

and $\|\dot{x}\|$ is thus a quasi-norm.

If $\{\dot{x}_n\}$ is a Cauchy sequence in E/N, we can suppose, by passing to a sub-sequence if necessary, that $\|\dot{x}_{n+1}-\dot{x}_n\|<2^{-n}$. Then we can inductively select elements $x_n\in\dot{x}_n$ such that $\|x_{n+1}-x_n\|<2^{-n}$. Since E is complete, the Cauchy sequence $\{x_n\}$ has a limit x_0, and if $\dot{x}_0$ is the coset containing x_0, then $\|\dot{x}_n-\dot{x}_0\|\leq\|x_n-x_0\|$ so that $\{\dot{x}_n\}$ has the limit $\dot{x}_0$. If a Cauchy sequence has a convergent sequence, then the original sequence is convergent. Thus E/N is a (QN) space with the power r if E is a (QN) space with the power r.

In general, metric spaces are not necessary complete. But every incomplete metric space will be extended to a complete space. In the case of a quasi-normed space with the power r, we may also prove the completion theorem.

Theorem. *If E is a quasi-normed space with the power r, then the space may be regarded as a dense subspace of a (QN) space $\hat{E}$ with the power r.*

Proof. Let two Cauchy sequences, (x_n) and (y_n) in E be called equivalent if $\lim_{n\to\infty}\|x_n-y_n\|=0$. It is clear that this is indeed an equivalence relation, and that a class of equivalent sequences either all converges to the same point of E or does not converge at all. In the latter case the class of equivalent sequences determines a hole. We now define a new point with every hole. Let $\hat{E}$ be the set of old (all points of E) and new points. After this, let us denote an old point by x, a new one by y and any one of $\hat{E}$ by z.

Let z_1 and z_2 be two points of $\hat{E}$, and (x_{in}) be, for $i=1$ and 2, a sequence converging to z_i.

The sequence of numbers $\|x_{1n}-x_{2n}\|$ converges to a limit as $n\to\infty$, for

$$\begin{aligned} \big| \|x_{1m}-x_{2m}\|-\|x_{1n}-x_{2n}\| \big| &\le \big| \|x_{1m}-x_{2m}\|-\|x_{1m}-x_{2n}\| \big| \\ &\quad +\big| \|x_{1m}-x_{2n}\|-\|x_{1n}-x_{2n}\| \big| \\ &\le \|x_{2m}-x_{2n}\|+\|x_{1m}-x_{1n}\|\to 0, \end{aligned}$$

as $m, n\to\infty$. This limit is unchanged by replacing (x_{in}), for $i=1$ and 2, by any equivalent sequence. Therefore the limit is a function of the points z_1 and z_2 and we may denote it by $\|z_1-z_2\|_0$.

$\|z\|_0$ satisfies the conditions for a quasi-norm with the power r.

$$1^\circ \quad \|z_1+z_2\|_0=\lim_{n\to\infty}\|x_{1n}+x_{2n}\|\le\lim_{n\to\infty}(\|x_{1n}\|+\|x_{2n}\|)$$
$$=\lim_{n\to\infty}\|x_{1n}\|+\lim_{n\to\infty}\|x_{2n}\|=\|z_1\|_0+\|z_2\|_0$$

$$2^\circ \quad \|\lambda z\|_0=\lim_{n\to\infty}\|\lambda x_n\|=\lim_{n\to\infty}|\lambda|^r\|x_n\|=|\lambda|^r\lim_{n\to\infty}\|x_n\|=|\lambda|^r\|z\|_0$$

3° $\|z\|_0=0$, that is, $\lim_{n\to\infty}\|x_n\|=0$ if and only if $x_n=0$ as $n\to\infty$, therefore it must be $z=0$.

If x_1 and x_2 are old points, $\|x_1-x_2\|=\|x_1-x_2\|_0$ since $\|x_{1n}-x_{2n}\|\to\|x_1-x_2\|$. It has thus been shown that $\|z\|_0$ determines a quasi-normed space $\hat{E}$ with the power r in which E is contained isometrically.

If (x_n) is a Cauchy sequence of old points defining the new point y, then $x_n\to y$ in $\hat{E}$, for since $(x_n, x_n, \cdots)$ is one of the sequences converging to x_n in E and

$$\|x_n-y\|_0=\lim_{m\to\infty}\|x_n-x_m\|\underset{n}{\to}0.$$

Thus E is dense in $\hat{E}$.

Let (z_n) be any Cauchy sequence in $\hat{E}$, and x_n an old point such that $\|z_n-x_n\|_0<\frac{1}{n}$. Then

$$\begin{aligned} \|x_m-x_n\| &= \|x_m-x_n\|_0 \\ &\le \|x_m-z_m\|_0+\|z_m-z_n\|_0+\|z_n-x_n\|_0 \\ &\to 0 \end{aligned}$$

as $m, n\to\infty$. Thus (x_n) is a Cauchy sequence in E. Let z be the corresponding point of (x_n) in $\hat{E}$. Then

$$\|z_n-z\|_0\le\|z_n-x_n\|_0+\|x_n-z\|_0\to 0$$

since $x_n\to z$ in $\hat{E}$. Thus the sequence (z_n) is convergent in the space $\hat{E}$. Therefore the space $\hat{E}$ is complete and is a (QN) space.

References

[1] M. Pavel: On quasi normed spaces, Bull. Acad. Polon. Sci., Cl. III, **5**, no. 5, 479–484 (1957).

[2] S. Rolewicz: On a certain class of linear metric spaces, Bull. Acad. Polon. Sci., Cl. III, **5**, no. 5, 471–473 (1957).