105. *A Unique Continuation Theorem of a Parabolic Differential Equation*

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1. Introduction. Let G be a convex domain of the euclidean $n+1$-space $R_{t,x}$ $(-\infty<t<+\infty, -\infty<x_i<+\infty$ $(i=1,2,\cdots,n))$, containing a curve C: $\{(t, x_i(t))\,|\,t\in[a,b]\}$, where $x_i(t)\in C^2[a,b]$.

Consider real solutions u of an inequality of the following kind:

$$(1.1)\qquad \left|\frac{\partial u(t,x)}{\partial t}-a_{ij}(t,x)\frac{\partial^2 u(t,x)}{\partial x_i\partial x_j}\right|\leqq M\left\{\sum_1^n\left|\frac{\partial u(t,x)}{\partial x_i}\right|+|u(t,x)|\right\}.$$

Here $((a_{ij}(t,x)))$ denotes a positive definite, symmetric matrix of real valued functions $a_{ij}(t,x)\in C^2(G)$, and M a constant.

Our purpose in this note is to prove the following theorem for solutions of (1.1).

Theorem. If u is a solution of (1.1) in the convex domain G and if for any $\alpha>0$,

$$(1.2)\qquad \lim_{r\to 0}\max_{\substack{|x-x(t)|=r\\ t\in[a,b]}}\left\{|u(t,x)|,\left|\frac{\partial u}{\partial t}(t,x)\right|,\left|\frac{\partial u}{\partial x_i}(t,x)\right|,\left|\frac{\partial^2 u}{\partial x_i\partial x_j}\right|\right\}|x-x(t)|^{-\alpha}=0$$

then u vanishes identically in the horizontal component.

The method is based upon the ideas of H. O. Cordes [2] and E. Heinz [3]. The tools used are all elementary, but our proof is somewhat complicated.

2. The Cordes' transformation. Assuming $[a,b]\supset[-\varepsilon,1+\varepsilon]$ $(\varepsilon>0)$, let $\mathring{A}(t)$ be the positive square root of the matrix $A(t)=((a_{ij}(t,x(t))))$. Let

$$x-x(t)=\mathring{A}(t)\tilde{x}\qquad\text{for}\quad t\in[-\varepsilon,1+\varepsilon],$$

then we may assume that for some $R_1>0$,

a) $a_{ik}(t,\tilde{x})\in C^2([-\varepsilon,1+\varepsilon]\times D_{R_1})$ $(D_{R_1}=\{x\,|\,|x|\leqq R_1\})$,

b) $a_{ik}(t,0)=\delta_{ik}$,

c) there are positive numbers C_1 and C_2 such that for any real vector $(\xi_1,\xi_2,\cdots,\xi_n)$

$$C_1\sum_1^n\xi_i^2\leqq\sum a_{ij}(t,\tilde{x})\xi_i\xi_j\leqq C_2\sum_1^n\xi_i^2.$$

From (a), (b) and (c) we see the following

Lemma 1. For some $R_2, \tilde{R}_2<R_1$ there is a topological transformation from $[-\varepsilon,1+\varepsilon]\times D_{R_2}$ onto $[-\varepsilon,1+\varepsilon]\times D_{\tilde{R}_2}$:

$$\tilde{y}=\tilde{y}(t,\tilde{x}),\ t=t$$

such that it satisfies the following conditions:

I. 1) $\tilde{y}(t,0)\equiv 0$,

2) $\dfrac{\partial\tilde{y}_i}{\partial\tilde{x}_j}, \dfrac{\partial\tilde{x}_i}{\partial\tilde{y}_j}, \dfrac{\partial^2\tilde{y}_i}{\partial\tilde{y}_j\partial\tilde{x}_k}, \dfrac{\partial^2\tilde{x}_i}{\partial\tilde{y}_j\partial\tilde{y}_k}$ are continuous over $[-\varepsilon, 1+\varepsilon]\times(D_{\tilde{R}_2}-\{0\})$ and

$$\left|\frac{\partial\tilde{y}_i}{\partial\tilde{x}_j}\right|<C, \left|\frac{\partial\tilde{x}_i}{\partial\tilde{y}_j}\right|<C, \left|\frac{\partial^2\tilde{y}_i}{\partial\tilde{x}_j\partial\tilde{x}_k}\right|<C|y|^{-1}, \left|\frac{\partial^2\tilde{x}_i}{\partial\tilde{y}_j\partial\tilde{y}_k}\right|<C|y|^{-1},$$

3) $\dfrac{\partial\tilde{y}_i}{\partial t}$ is continuous over $[-\varepsilon, 1+\varepsilon]\times(D_{\tilde{R}_2}-\{0\})$ and $\left|\dfrac{\partial\tilde{y}_i}{\partial t}\right|<C$,

II. for any $\tilde{y}:0<|\tilde{y}|\leqq\tilde{R}_2$, there is a suitable polar coordinates (r,φ_σ) such that

$$(2.1)\qquad \frac{\partial}{\partial\tilde{x}_i}a_{ij}(t,\tilde{x})\frac{\partial u}{\partial\tilde{x}_j}=p(t,\tilde{y})\Big(\frac{\partial^2}{\partial r^2}+\frac{n-1}{r}\frac{\partial}{\partial r}+\frac{N}{r^2}\Big)u+p_i(t,\tilde{y})\frac{\partial u}{\partial\tilde{y}_i},$$

where $p(t,\tilde{y})$, $p_i(t,\tilde{y})$ and the operator N satisfy the following conditions:

1. $C>p(t,\tilde{y})>C^{-1}$, $|p_i(t,\tilde{y})|<C$,

2. $|p(t,\tilde{y})|<C$, $\left|\dfrac{\partial p(t,\tilde{y})}{\partial t}\right|<C$, $\left|\dfrac{\partial p(t,\tilde{y})}{\partial r}\right|<C$, $\left|\dfrac{\partial p(t,\tilde{y})}{\partial\varphi_\sigma}\right|<C$,

3. $N=\dfrac{1}{\lambda(\tilde{y})}\dfrac{\partial}{\partial\varphi_\sigma}\lambda(\tilde{y})\bar{a}_{\sigma\tau}(t,\tilde{y})\dfrac{\partial}{\partial\varphi_\tau}$, $\lambda(y)=\dfrac{dO_1}{d\varphi_1 d\varphi_2\cdots d\varphi_{n-1}}$,

where dO_1 is the usual surface element of the unit sphere,

4. there are two positive numbers $\bar{C}_1$ and $\bar{C}_2$ such that

$$\bar{C}_1\sum_1^{n-1}\eta_\sigma^2\leqq\sum\bar{a}_{\sigma\tau}(t,\tilde{y})\eta_\sigma\eta_\tau=\bar{C}_2\sum_1^{n-1}\eta_\sigma^2$$

for any real vector $\{\eta_1\cdots\eta_{n-1}\}$,

5. $\bar{a}_{\sigma\tau}$, $\dfrac{\partial\bar{a}_{\sigma\tau}}{\partial t}$, $\dfrac{\partial\bar{a}_{\sigma\tau}}{\partial r}$ and $\dfrac{\partial\bar{a}_{\sigma\tau}}{\partial\varphi_\rho}$ are continuous and

$$|\bar{a}_{\sigma\tau}|<C, \left|\frac{\partial\bar{a}_{\sigma\tau}}{\partial t}\right|<C, \left|\frac{\partial\bar{a}_{\sigma\tau}}{\partial r}\right|<C, \left|\frac{\partial\bar{a}_{\sigma\tau}}{\partial\varphi_\rho}\right|<C,$$

where the constants $\bar{C}_1$, $\bar{C}_2$ and C depend only on R_1, C_1, C_2 and the derivatives of $a_{ij}(t,x)$ of order $\leqq 2$. (Here we use a finite number of fixed, suitable systems of polar-coordinates covering the unit sphere.)

To prove the above proposition, we only remark that

$$\nu_\sigma(t,r,\theta_1,\theta_2,\cdots,\theta_{n-1})=\frac{\sum a_{ik}(t,\tilde{x})\dfrac{\tilde{x}_i}{r}\cdot\theta_{\sigma|\tilde{x}_k}}{\sum a_{ik}(t,\tilde{x})\dfrac{\tilde{x}_i}{r}\cdot\dfrac{\tilde{x}_k}{r}}$$

satisfies the following conditions: for any $t\in[-\varepsilon,1+\varepsilon]$ and $\tilde{x}:0\leqq|\tilde{x}|<R_1$ the function $\nu_\sigma(t,r,\theta)$, $\nu_{\sigma|r}$, $\nu_{\sigma|\theta_\tau}$, $\nu_{\sigma|t}$, $\nu_{\sigma|\theta_\tau\theta_\rho}$, $\nu_{\sigma|r,\theta_\tau}$ and $\nu_{\sigma|t,\theta_\tau}$ are all continuous, and for any $t\in[-\varepsilon,1+\varepsilon]$ and $\tilde{x}:0<|\tilde{x}|\leqq R_1$, $\nu_{\sigma|rr}$, $\nu_{\sigma|rt}$ and $\nu_{\sigma|tt}$ are continuous, where $\nu(t,r,\theta)$ is considered as a function of t, r, θ. Here and in the proof of the following sections $u|_h$ denotes $\dfrac{\partial u}{\partial h}$.

Furthermore by the transformation: $(t, \tilde{y})=(t, r, \varphi_\sigma)\rightarrow(t, s, \varphi_\sigma)=(t, y)$:

$$s(r)=re^{\int_0^r (e^{-m_0\tau}-1)\frac{d\tau}{\tau}}$$

we see the following

Lemma 2. By the transformation $(t, \tilde{y})\rightarrow(t, y)$ with a sufficiently large m_0, the following condition is satisfied: for any $w\in C^2(y:|y|=1)$ and for any $t\in[-\varepsilon, 1+\varepsilon]$,

III. $\frac{\partial}{\partial s}\int N\omega\cdot\omega dO_1 \leqq m_0\int N\omega\cdot\omega dO_1<0$

as well as Conditions I and II.

3. The first inequality. Using the above lemmas, we will deduce the Heinz' inequality with respect to (1.1). For this purpose we may assume that

$$(3.1)\qquad \begin{aligned} L_1(u)&=q(t, x)\frac{\partial u}{\partial t}-a_{ij}(t, x)\frac{\partial^2 u}{\partial x_i\partial x_j}+b_i(t, x)\frac{\partial u}{\partial x_i}\\ &=q(t, x)\frac{\partial u}{\partial t}-\left(\frac{\partial^2}{\partial r^2}+\frac{n-1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}N\right)u, \end{aligned}$$

where $q(t, x)(>\delta>0)\in\bar{C}^1(t, r, \varphi_\sigma)$, $a_{ij}(t, x)\in\bar{C}^0(t, x)$, $b_i(t, x)\in\bar{C}^0(t, x)$ and the coefficients of $N\in\bar{C}^1(t, r, \varphi_\sigma)$ $(0<r\leqq R)$ for fixed, suitable polar coordinates (r, φ_σ) of x.

Furthermore we may assume that u satisfies the condition (1.2) with $x_i(t)=0$ for $t\in[-\varepsilon, 1+\varepsilon]$.
Put $D_{r_0, K_0}=\{(t, x)\,|\,0\leqq t\leqq 1$ and $|x|\leqq r_0\wedge K_0^{-1}t\}$ and let $\varphi_{r_0, K_0}(t, x)$ be such that: (1) it is in $\bar{C}^2(D_{r_0, K_0}-\{0\})$, (2) its carrier is contained in D_{r_0, K_0}, (3) $\varphi_{r_0, K_0}\equiv 1$ in $D_{\frac{1}{2}r_0, 2K_0}-\{0\}$ and (4) $v=u\,\Psi_{r_0, K_0}$ also satisfies the condition (1.2).

Furthermore let f be a monotone decreasing, smooth function such that

$$f(t)=1 \quad\text{for}\quad t\leqq\frac{2}{3},\qquad f(t)>0\quad\text{for}\quad t<1\quad\text{and}\quad f(1)=0.$$

Let $\alpha(t)=\alpha f(t)+(n-2)$. Finally let $\varphi(t)$ be a monotone increasing, smooth function such that

$$\varphi(t)=t\quad\text{for}\quad t\leqq\frac{1}{4},\qquad \varphi(t)=1\quad\text{for}\quad t\geqq\frac{1}{2}$$

and let $\Phi_\alpha(t)=\varphi(t)^{2\alpha}e^{kt}$. Then we see the following

Lemma 3. For sufficiently small r_0 and sufficiently large K_0 and k there is a constant α_0 such that for any $\alpha>\alpha_0$,

$$(3.2)\qquad \begin{aligned} &\alpha^2kK_1\iint_{D_{r_0, K_0}}|v|^2r^{-\alpha(t)}\Phi_\alpha(t)dxdt\\ &\leqq\iint_{D_{r_0,K_0}}|L_1(v)|^2r^{2-\alpha(t)}\Phi_\alpha(t)dxdt+\alpha^2K_2\int_{D_{r_0}}|v|^2r^{-\alpha(t)}\Phi_\alpha(t)dx\Big|_{t=1}, \end{aligned}$$

where K_0, K_1, K_2 are constant numbers depending only the derivatives with respect to t, r, and φ_σ, of q of order $\leqq 1$, the derivatives with respect to t, r of the coefficients of N, and f of order $\leqq 1$, which are independent of systems of polar-coordinates $\{\varphi_\rho\}$. (Here and in the following proofs we denote such constants by K.)

(Outline of the proof). By the usual limit processes [1, 2] we may assume that the coefficients of L_1 and v are sufficiently smooth. Let $\beta(t)=\frac{1}{2}(\alpha(t)-n+2)$ and $u=r^{\beta(t)}z$. Then we see that

$$\begin{aligned}(3.3)\quad &\iint |L_1(v)|^2 r^{2-\alpha(t)}\Phi_\alpha(t)dxdt\\ &\geqq \iiint \{|qr^2z_{|t}|^2+|L^{**}z|^2+2L^*z\cdot L^{**}z-2r^2z_{|t}\cdot q\cdot(L^*z+L^{**}z)\}\cdot\\ &\quad\cdot r^{-1}\Phi_\alpha(t)dO_1drdt,\end{aligned}$$

where

$$L^*z=r(rz_{|r})_{|r}+Nz+\left[\frac{\alpha^2-(n-2)^2}{4}-q\alpha\frac{f'(t)}{2}r^2\log r\right]z,$$

$$L^{**}z=\alpha rz_{|r}.$$

From $\varphi'\geqq 0$ it implies that for any K there is a number k_0 such that for $k>k_0$

$$(q\Phi_\alpha)_{|t}-K(q\Phi_\alpha)\geqq\frac{1}{2}(q\Phi_\alpha)_{|t}.$$

Therefore by partial integrations with respect to t and r and from III in §2 and the relation $f'\leqq 0$, it follows that

$$\begin{aligned}(3.3)\quad \geqq &\iiint \{r^3q^2\Phi_\alpha(z_{|t})^2+\alpha^2r\Phi_\alpha(z_{|r})^2-2(\alpha-2)r^2q\Phi_\alpha z_{|r}\cdot z_{|t}\\ &+2r^3(q\Phi_\alpha)_{|r}z_{|r}\cdot z_{|t}-r^3(q\Phi_\alpha)_{|t}(z_{|r})^2-\alpha K_3r^2(q\Phi_\alpha)_{|t}z^2\}dO_1drdt\\ &-\alpha^2K_4\iiint r\Phi_\alpha\cdot z^2dO_1drdt\\ &+\iiint\left\{-\alpha m_0\Phi_\alpha\cdot z\cdot Mz+r(q\Phi_\alpha)_{|}z\cdot Mz+rq\Phi_\alpha z\cdot M_{|t}z\right\}dO_1drdt\\ &+\iint\Big\{qr\Phi_\alpha\cdot z\cdot r(rz_{|r})_{|r}+2r(q\Phi_\alpha)z^2-r^3(q\Phi_\alpha)_{|r}z\cdot z_{|r}\\ &+(q\Phi_\alpha)_{|r}r^2z^2-rq\Phi_\alpha\cdot z\cdot Mz+rq\Phi_\alpha\cdot\frac{\alpha^2-(n-2)^2}{4}\cdot z^2\\ &-\alpha r^2q\Phi_\alpha\cdot K_5z^2\Big\}dO_1dr\Big|_{t=1}.\end{aligned}$$

Furthermore we note that for sufficiently small r_0, for sufficiently large K_0 and k, there is a number α_0 such that for any $\alpha>\alpha_0$

$$(3.4)_1\qquad \alpha\Phi_\alpha-|r^2(q\Phi_\alpha)_{|t}|\geqq 0,$$

$$(3.4)_2\qquad m_0\alpha\Phi_\alpha-|r(q\Phi_\alpha)_{|t}|-rq\Phi_\alpha K\geqq 0.$$

From $(3.4)_1$, $(3.4)_2$ and II in §2, it follows that

$$(3.3)\quad \geqq K_6\alpha^2k\iiint rz^2\Phi_\alpha dO_1drdt-K_7\alpha^2\iint rz^2\Phi_\alpha dO_1dr\Big|_{t=1},$$

which implies (3.1).

4. The second inequality. Let r_0 and K_0 be fixed numbers such that for sufficiently large k and α_0, (3.2) is valid.

Then using the relation: $f(1)=0$, $f(t)>0$ for $t<1$ and $\varphi(t)=1$ for $t\geqq\frac{1}{2}$,

we see that even if $\int|v|^2dx\Big|_{t=1}\neq 0$, there is an interval $[c, d]$ $\left(\frac{1}{2}<c<d<1\right)$ such that for any k and for any α $(>\alpha_0(k, u))$

$$\int|v|^2r^{-\alpha(t)}\Phi_\alpha(t)dx\Big|_{t=1}\leqq\int|v|^2r^{-\alpha(t)}\Phi_\alpha(t)dx\Big|_t (t\in[c, d]).$$

Therefore from (3.2) it follows that for sufficiently large k there is a constant K_8 and α_0 such that for $\alpha>\alpha_0$

$$\alpha^2K_8k\iint|v|^2r^{-\alpha(t)}\Phi_\alpha(t)dxdt\leqq\iint|L_1(v)|^2r^{2-\alpha(t)}\Phi_\alpha(t)dxdt. \tag{4.1}$$

Then from (3.1), (4.1) and $(3.4)_1$ we see the following

Lemma 4. For sufficiently small r_0 and for sufficiently large K_0 and $k>k_0$, there are constants K_9 and α_0 such that for $\alpha>\alpha_0$

$$\begin{aligned}&\iint\left(\frac{\alpha^2}{r_0^2}|v|^2+|v_{|x_i}|^2\right)r^{2-\alpha(t)}\Phi_\alpha(t)dxdt\\ &\leqq k^{-\frac{1}{2}}K_9\iint|L_1(v)|^2r^{2-\alpha(t)}\Phi_\alpha(t)dxdt,\end{aligned} \tag{4.2}$$

where k_0 depends on u and K.

5. The proof of Theorem. In this section we use the notations in §1 and §2. By (1.1) we may assume that for some r_0, ε

$$|L_1(u)|\leqq M\left\{|u|+\sum_1^n\left|\frac{\partial u}{\partial y_i}\right|\right\}\quad\text{for}\quad t\in[-\varepsilon, 1+\varepsilon]\quad\text{and}\quad r\leqq r_0 \tag{5.1}$$

where $2K_0r_0<\frac{1}{4}$.

Then from Lemma 4 we see that for any $\alpha(>\alpha_0(K, r_0, k))$

$$\begin{aligned}&\iint_{D_{r_0/2,2K_0}}(|u|^2+|u_{|y_i}|^2)r^{2-\alpha(t)}\Phi_\alpha dydt\\ &\leqq k^{-\frac{1}{2}}K\iint_{D_{r_0,K_0}}|L_1(v)|^2r^{2-\alpha(t)}\Phi_\alpha dydt\\ &\leqq k^{-\frac{1}{2}}K\iint_{D_{r_0,K_0}-D_{r_0/2,2K_0}}|L_1(v)|^2r^{2-\alpha(t)}\Phi_\alpha dydt\\ &\qquad+k^{-\frac{1}{2}}K\cdot M\iint_{D_{r_0/2,2K_0}}(|u|^2+|u_{|y_i}|^2)r^{2-\alpha(t)}\Phi_\alpha dydt.\end{aligned}$$

Accordingly choosing k sufficiently large such that

$$2K\cdot M<k^{\frac{1}{2}},$$

it follows that for any $\alpha>\alpha_0$

$$\frac{1}{2}\left(\frac{r_0}{3K_0}\right)^{2-\alpha-(n-2)} \iint\limits_{\frac{1}{4}\leqq t\leqq\frac{2}{3},\, r\leqq r_0/3K_0} \{|u|^2+|u_{|y_i}|^2\}e^{kt}dydt$$
$$\leqq\left(\frac{r_0}{2}\right)^{2-\alpha-(n-2)} k^{-\frac{1}{2}}K \iint\limits_{t\geqq 2K_0\cdot r_0,\, D_{r_0,K_0}-D_{r_0/2,2K_0}} |L_1(v)|^2 e^{kt}dydt$$
$$+\left(\frac{1}{2K_0}\right)^{-\alpha-n} k^{-\frac{1}{2}}K \iint\limits_{t\leqq 2K_0\cdot r_0,\, D_{r_0,K_0}-D_{r_0/2,2K_0}} |L_1(v)|^2 e^{kt}dydt.$$

Therefore tending $\alpha\to\infty$, we see that

$$u(t, y)=0 \quad \text{for} \quad t\in\left[\frac{1}{4},\frac{2}{3}\right], \qquad r\leqq r_0/3K_0.$$

Since, in the above proof, the numbers $\left\{\varepsilon, \frac{1}{4}\right\}$ and $\frac{2}{3}$ may be replaced by arbitrary small and large numbers respectively, we see that $u(t,x)=0$ in a neighbourhood of C in $(a,b)\times R_x$. Then by a topological argument and from Lemma 1 and Lemma 4 also, we see that $u(t,x)\equiv 0$ in the horizontal component stated in §1.

Another detailed proof of Theorem and the results in my previous paper [4] with other consequences will be published in the Osaka Mathematical Journal next year.

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