101. Purely Algebraic Characterization of Quasiconformality

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1. Consider two Riemann surfaces R and R'. Assume the existence of a quasiconformal mapping[1] T of R onto R' in the sense of Pfluger-Ahlfors-Mori [6,1,3]. In this case we say that R and R' are *quasiconformally equivalent*. In particular, if the maximal dilatation $K(T)$ (of T (cf. [1]) is 1), R and R' are said to be *conformally equivalent*.

This note will communicate a certain criterion of quasiconformal equivalence in terms of function algebras, details of which will be published later.

2. Let R be a Riemann surface and $M(R)$ be *Royden's algebra* [8, 4] associated with R, i.e. the totality of complex-valued bounded a.c.T.[2] functions on R with finite Dirichlet integrals over R. The algebraic operations are defined as follows: $(f+g)(p)=f(p)+g(p)$, $(f\cdot g)(p)=f(p)\cdot g(p)$ and $(\alpha\cdot f)(p)=\alpha f(p)$. Then $M(R)$ is a commutative algebra over the complex number field.

3. As an improvement of the author's previous result [4], we mention the following algebraic criterion of quasiconformal equivalence:

Theorem 1. *Two Riemann surfaces R and R' are quasiconformally equivalent if and only if $M(R)$ and $M(R')$ are algebraically isomorphic.*

4. Royden's algebra $M(R)$ can be *normed* by the following:

$$\|f\|=\sup_R|f|+\left(\iint_R df\wedge{}^*\overline{df}\right)^{1/2}.$$

As a special case of Theorem 1 and as an improvement of [5], we get the following normed algebraic criterion of conformal equivalence:

Theorem 2. *Two Riemann surfaces R and R' are conformally equivalent if and only if $M(R)$ and $M(R')$ are isometrically isomorphic.*

5. Theorems 1 and 2 follow from the following more precise facts.

Let $Q(R,R')$ be the totality of quasiconformal mappings of R onto R' and $I(R,R')$ be the totality of algebraic isomorphism of $M(R)$ onto $M(R')$. Then there exists a one-to-one correspondence $T\leftrightarrow\sigma$ between $Q(R,R')$ and $I(R,R')$. This correspondence is given by $f^\sigma=f\circ T^{-1}$

1) Including both direct and indirect ones.

2) Abbreviation of "absolutely continuous in the sense of Tonelli". For the definition, refer to [7, 9, 10].

$(f \in M(R))$. Moreover it holds

$$1 \leqq K^*(T) \leqq \| \sigma \|^2 \leqq K(T),$$

where

$$K^*(T) = \inf \{\lambda;\ \lambda^{-1} \operatorname{mod} A \leqq \operatorname{mod} TA \leqq \lambda \operatorname{mod} A\},$$ [3]

where A runs over all annuli on R, and where

$$\| \sigma \| = \inf \{\lambda;\ \lambda^{-1} \| f \| \leqq \| f^\sigma \| \leqq \lambda \| f \| \text{ for all } f \text{ in } M(R)\}.$$ [4]

6. The proof of the fact mentioned in §5 is divided into two parts. The one part deals with purely topological algebraic matters and the other part is concerned with the local theory of quasiconformal mappings.

7. The first part of the proof of §5 is to find a topological mapping T of R onto R' such that $f^\sigma = f \circ T^{-1}$ for a given σ in $I(R, R')$.

To this end we state some general theorem. Let $\mathfrak{S} = \{\Omega\}$ be the totality of locally compact Hausdorff spaces Ω. Let $\mathfrak{C}$ be the totality of the pairs (f, Ω) where $\Omega \in \mathfrak{S}$ and f be a complex-valued continuous function on Ω. Let $\mathfrak{P}$ be a subfamily of $\mathfrak{C}$. We denote by $\mathfrak{P}(\Omega)$ the set $\{f;\ (f, \Omega) \in \mathfrak{P}\}$. We consider the following conditions for a subfamily $\mathfrak{P}$ of $\mathfrak{C}$:

(1) $\mathfrak{P}(\Omega)$ *is an algebra over the complex number field with* 1;

(2) $\mathfrak{P}(\Omega)$ *separates points strongly*, i.e. for an open neighborhood V of any point p in Ω, there exists a function f in $\mathfrak{P}(\Omega)$ such that $f = 0$ on $\Omega - V$ and 1 near p in V;

(3) $\mathfrak{P}(\Omega)$ *is inverse-closed*, i.e. if f is in $\mathfrak{P}(\Omega)$ and $\inf |f| > 0$, then $1/f$ belongs to $\mathfrak{P}(\Omega)$;

(4) $\mathfrak{P}(\Omega)$ *is self-adjoint*, i.e. $f \in \mathfrak{P}(\Omega)$ implies $f^* \in \mathfrak{P}(\Omega)$, where $f^*(p) = \overline{f(p)}$ (complex conjugate);

(5) $f \in \mathfrak{P}(\Omega)$ *is bounded*;

(6) $\mathfrak{P}$ *is monotone*, i.e. if Ω and Ω' are in $\mathfrak{S}$ and Ω' is an open subset of Ω' and if $(f, \Omega) \in \mathfrak{P}$ and f vanishes on a neighborhood of the relative boundary of Ω with respect to Ω', then $(f', \Omega') \in \mathfrak{P}$, where $f' = f$ on Ω and $f' = 0$ on $\Omega' - \Omega$.

We say that a topological mapping T of Ω onto Ω' has *property* $\mathfrak{P}$ if $(f \circ T^{-1}, \Omega') \in \mathfrak{P}$ and $(f, \Omega) \in \mathfrak{P}$ are equivalent. An algebraic isomorphism σ of $\mathfrak{P}(\Omega)$ onto $\mathfrak{P}(\Omega')$ is said *to be induced by a topological mapping T locally* if T is a topological mapping of an open subset Ω_1 of Ω onto an open subset Ω_1' of Ω' and $f^\sigma(Tp) = f(p)$ for p in Ω_1. By using these terminologies, our additional condition is stated as follows:

(7) *If an algebraic isomorphism of $\mathfrak{P}(\Omega)$ onto $\mathfrak{P}(\Omega')$ is induced*

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3) An annulus A is conformally equivalent to a circular ring $1 < |z| < e^\mu$. The uniquely determined number μ is denoted by $\operatorname{mod} A$.

4) It is known that if one of $K^*(T)$, $\| \sigma \|$ and $K(T)$ is 1, then T is a conformal mapping [5].

by a topological mapping T locally, then T has property $\mathfrak{P}$.

We shall say that a subfamily $\mathfrak{P}$ of $\mathfrak{C}$ satisfying the conditions (1)–(7) is an *admissible subfamily* of $\mathfrak{C}$.

A point p in Ω is said to be *$\mathfrak{P}$-removable* if for any $(f, \Omega-p)$ in $\mathfrak{P}$, we can continue f to p so as to be $(f, \Omega)\in\mathfrak{P}$. If Ω has no $\mathfrak{P}$-removable point, we write $\Omega\in\mathfrak{S}_{\mathfrak{P}}$.

Then our theorem is stated as follows:

Theorem 3. *Let $\mathfrak{P}$ be an admissible subfamily of $\mathfrak{C}$ and, Ω and Ω' be in $\mathfrak{S}_{\mathfrak{P}}$. Assume that σ is an algebraic isomorphism of $\mathfrak{P}(\Omega)$ onto $\mathfrak{P}(\Omega')$. Then there exists a topological mapping T of Ω onto Ω' such that $f^{\sigma}=f\circ T^{-1}$.*

8. In order to apply this theorem to our Royden's algebra, we consider the subfamily M of $\mathfrak{C}$ such that (f, Ω) belongs to M if Ω is an open subset of a Riemann surface and f is a bounded a.c.T. function on Ω with finite Dirichlet integral over Ω.

It is verified that M satisfies (1)–(6) and any Riemann surface belongs to $\mathfrak{S}_{M}$. Applying Gelfand's theorem [2], we see that M satisfies (7).

9. Thus we have found a topological mapping T of R onto R' such that $f^{\sigma}=f\circ T^{-1}$ for given $\sigma\in I(R, R')$. The second part of the proof of §5 is to show that $T\in Q(R, R')$. To this end we first notice $\|\sigma\|<\infty$, which follows from the completeness of $M(R)$ and $M(R')$ and from Gelfand's theorem [2], and next we prove the relation $K^{*}(T)\leqq\|\sigma\|^{2}$. Applying the result obtained in [4] and [5], we conclude that T belongs to $Q(R, R')$.

10. Let $M^{n}(R)$ $(0\leqq n\leqq\infty)$ be the totality of C^{n} functions in $M(R)$ $(=M^{0}(R))$. Corresponding to Theorems 1 and 2, we can show the followings:

Theorem 4. *Two Riemann surfaces R and R' are quasiconformally equivalent if and only if $M^{n}(R)$ and $M^{n}(R')$ are algebraically isomorphic for some n and hence for all $n=0,1,2,\cdots,\infty$.*

Theorem 5. *Two Riemann surfaces R and R' are conformally equivalent if and only if $M^{n}(R)$ and $M^{n}(R')$ are isometrically isomorphic for some n and hence for all $n=0,1,2,\cdots,\infty$.*

References

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