97. *Note on Left Simple Semigroups*

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1. Teissier [5] considered homomorphisms of a left simple semigroup with no idempotent onto a semigroup which contains at least one idempotent, and characterized the inverse images of idempotents in such homomorphic mappings. In this note, we consider a method of constructing such a homomorphism, which turns out to be the finest in such homomorphisms.

We use terminologies in [2] without definitions and use the results obtained in [2] and [3] freely.

2. In this note, we denote a left simple semigroup by S.

In S, we define a binary relation ${}_S\Sigma$ as follows:

for $a, b \in S$, $a \equiv b({}_S\Sigma)$ means that there exists a finite sequence of elements $m_1, \cdots, m_{n-1}$ such that

$$aS \between m_1S \between \cdots \between m_{n-1}S \between bS,$$

where $xS \between yS$ signifies that the sets xS and yS have at least one element in common.

It is easy to see that ${}_S\Sigma$ is an equivalence relation in S, which is left regular, that is,

$$a \equiv b \ ({}_S\Sigma) \quad \text{implies} \quad ca \equiv cb \ ({}_S\Sigma).$$

In Dubreil's terminology, ${}_S\Sigma$ is the generalized left reversible equivalence associated to S [1, p. 258].

Lemma 1. *$ac \equiv a({}_S\Sigma)$ for all $a, c \in S$* (cf. [1, p. 260, Théorème 8]).

Proof. For any $s \in S$, we have $(ac)s = a(cs)$. Hence we have

$$acS \between aS, \text{ and so } ac \equiv a({}_S\Sigma).$$

Lemma 2. *${}_S\Sigma$ is an equivalence relation which is regular.*

Proof. It suffices to show that ${}_S\Sigma$ is right regular, that is, $a \equiv b({}_S\Sigma)$ implies $ac \equiv bc({}_S\Sigma)$. And, in fact, if $a \equiv b({}_S\Sigma)$, then, by Lemma 1, we have $ac \equiv a \equiv b \equiv bc({}_S\Sigma)$.

Now, we denote the core of S by I. I is a normal and left unitary subsemigroup of S [2, Theorem 1].

Lemma 3. *For any $a \in S$, there exists an element $i \in I$ such that $a \equiv i({}_S\Sigma)$.*

Proof. Since S is left simple, we can take an element u such that $ua = a$. u is clearly an element of I, and also, by Lemma 1, we have $a = ua \equiv u({}_S\Sigma)$.

Since ${}_S\Sigma$ is a regular equivalence relation in S, ${}_S\Sigma$ induces a regular equivalence relation in semigroup I. Hence we can consider the quotient semigroup $I/_S\Sigma$ of the semigroup I. We denote the totality of classes which are the elements of $I/_S\Sigma$ by $\{J_\lambda;\ \lambda\in\Lambda\}$, and denote the product of J_λ and J_μ in $I/_S\Sigma$ by $J_\lambda\cdot J_\mu$.

Lemma 4. *$J_\lambda\cdot J_\mu=J_\lambda$ for all J_λ, $J_\mu\in I/_S$.*

Proof. For $j\in J_\lambda$, $j'\in J_\mu$, $J_\lambda\cdot J_\mu$ is, by definition, the class which contains the element jj'. On the other hand, by Lemma 1, we have $jj'\equiv j({}_S\Sigma)$, and so jj' belongs to the class which contains j, that is, belongs to J_λ. Hence $J_\lambda\cdot J_\mu=J_\lambda$.

Now, we consider a mapping φ of S into $I/_S\Sigma$. For any $a\in S$, $\varphi(a)$ is defined to be the class J_λ which contains an element i such that $i\equiv a({}_S\Sigma)$. By Lemma 3, $\varphi(a)$ is defined certainly for all $a\in S$, and it is clear that $\varphi(a)$ is uniquely determined irrespective of the choice of element i.

Lemma 5. *$\varphi(a)$ is a homomorphism of S onto $I/_S\Sigma$.*

Proof. For $a, b\in S$, we take an element i such that $a\equiv i({}_S\Sigma)$. Then, by Lemma 1, we have $ab\equiv a\equiv i({}_S\Sigma)$, and so $\varphi(ab)=\varphi(a)$. Hence, by Lemma 4, we have $\varphi(ab)=\varphi(a)=\varphi(a)\cdot\varphi(b)$, and so φ is a homomorphism. On the other hand, for any $J_\lambda\in I/_S\Sigma$, we take an element $j\in J_\lambda$. Then, by definition, $\varphi(j)=J_\lambda$, and so φ is a homomorphism onto $I/_S\Sigma$.

In [2], we proved that, for a left simple semigroup S, there exists a homomorphism θ of S onto a group G such that the kernel of θ is I [2, Theorem 3].

Lemma 6. *If $\theta(a)=\theta(b)$ and $xa=b$, then $x\in I$.*

Proof. Under the assumption of this lemma, we have $\theta(x)\theta(a)=\theta(b)=\theta(a)$, and so $\theta(x)=e$, where e is the identity element of group G. Therefore we have $x\in I$.

Now, we consider the direct product T of $I/_S\Sigma$ and the group G above-mentioned. And we define a mapping ψ of S into T as follows:

$$\psi(a)=(\varphi(a), \theta(a)).$$

ψ is evidently a homomorphism of S into T.

Lemma 7. *An element (J_λ, g) of T is an idempotent, if and only if g is the identity element e of G.*

Proof. By Lemma 4, we have $(J_\lambda, g)(J_\lambda, g)=(J_\lambda\cdot J_\lambda, g^2)=(J_\lambda, g^2)$. Hence (J_λ, g) is an idempotent, if and only if $g^2=g$, and so if and only if $g=e$.

Lemma 8. $\psi^{-1}(J_\lambda, e)=J_\lambda$.

Proof. For any $j\in J_\lambda$, we have $\varphi(j)=J_\lambda$. Also, since $j\in J_\lambda\subseteq I$, we have $\theta(j)=e$. Hence we have $\psi(j)=(J_\lambda, e)$, and so $J_\lambda\subseteq\psi^{-1}(J_\lambda, e)$. Conversely, let us suppose that $\psi(a)=(J_\lambda, e)$. Then, since $\theta(a)=e$, we have $a\in I$. Hence $a\in J_\mu$ for some $\mu\in\Lambda$. But then, we have $\varphi(a)=J_\mu$ and $\psi(a)=(J_\mu, e)$. By assumption, we have $\psi(a)=(J_\lambda, e)$ and so $a\in J_\mu=J_\lambda$. Hence $\psi^{-1}(J_\lambda, e)\subseteq J_\lambda$.

Lemma 9. *ψ is a homomorphism onto T.*

Proof. Let (J_λ, g) be an arbitrary element of T. Since θ is a homomorphism onto G, there exists an element a such that $\theta(a)=g$. Also we take any element $j\in J_\lambda$. Then, by Lemma 8, we have $\psi(j)=(J_\lambda, e)$. Hence we have $\psi(ja)=\psi(j)\psi(a)=(J_\lambda, e)(\varphi(a), g)=(J_\lambda\cdot\varphi(a), eg)=(J_\lambda, g)$.

Summarizing the above lemmas, we obtain the following

Theorem 1. *ψ is a homomorphism of a left simple semigroup S onto a semigroup T which contains at least one idempotent, and in this homomorphism ψ, the inverse images of idempotents of T coincide with the classes J_λ which are the elements of $I/_S\Sigma$.*

3. Now we consider a homomorphism ψ^* of left simple semigroup S onto a semigroup T^* which contains at least one idempotent. Being a homomorphic image of a left simple semigroup, T^* is also left simple. And it is well known that left simple semigroup T^* with idempotent can be represented by the direct product of two semigroups U^* and G^*, where U^* is a right anti-semigroup in terminology of Thierrin [6], that is, a semigroup which satisfies the condition as follows:

$$uu'=u \qquad \text{for all} \quad u, u'\in U^*,$$

and G^* is a group (cf. [4]).

In association with the homomorphism ψ^*:

$$\psi^*(a)=(u, g) \qquad \text{where} \quad a\in S,\ u\in U^*,\ g\in G^*,$$

we consider mappings φ^* and θ^* such that

$$\varphi^*(a)=u, \qquad \theta^*(a)=g.$$

φ^* and θ^* are easily seen to be homomorphisms of S onto U^* and G^* respectively.

Lemma 10. *An element (u, g) of T^* is an idempotent, if and only if g is the identity element e^* of the group G^*.*

Proof is similar as in Lemma 7.

Lemma 11. *$\varphi^*(ab)=\varphi^*(a)$ for all $a, b\in S$.*

Proof. $\varphi^*(ab)=\varphi^*(a)\varphi^*(b)=\varphi^*(a)$.

Lemma 12. *$\varphi(a)=\varphi(b)$ implies $\varphi^*(a)=\varphi^*(b)$.*

Proof. If $\varphi(a)=\varphi(b)$, then the elements a and b are congruent modulo $_S\Sigma$ to an element of the class $\varphi(a)=\varphi(b)$, and so $a\equiv b(_S\Sigma)$. Hence there exists a finite sequence of elements $m_1, m_2, \cdots, m_{n-1}$ such that

$$aS\between m_1S\between m_2S\between\cdots\between m_{n-1}S\between bS.$$

Therefore $as=m_1s_1$ for some $s, s_1\in S$. But then, we have, by Lemma 11,

$$\varphi^*(a)=\varphi^*(as)=\varphi^*(m_1s_1)=\varphi^*(m_1).$$

Similarly, we have

$$\varphi^*(m_1)=\varphi^*(m_2), \cdots, \varphi^*(m_{n-1})=\varphi^*(b),$$

and hence we have $\varphi^*(a)=\varphi^*(b)$.

Lemma 13. *$\theta^*(i)=e^*$ for all $i\in I$.*

Proof. θ^* is a homomorphism of S onto group G^*. Hence the kernel $\{x; \theta^*(x)=e^*\}$ contains the core I [2, Theorem 2]. Hence, for any $i\in I$, we have $\theta^*(i)=e^*$.

Lemma 14. *$\theta(a)=\theta(b)$ implies $\theta^*(a)=\theta^*(b)$.*

Proof. Let us suppose that $\theta(a)=\theta(b)$. By the left simplicity of S, there exists an element x such that $xa=b$. Then, by Lemma 6, we have $x\in I$. Hence, by Lemma 13, we have $\theta^*(x)=e^*$, and so we have $\theta^*(b)=\theta^*(xa)=\theta^*(x)\theta^*(a)=\theta^*(a)$.

Lemma 15. *$\psi(a)=\psi(b)$ implies $\psi^*(a)=\psi^*(b)$.*

Proof. This is an immediate consequence of Lemmas 12 and 14.

By Lemma 15, we can consider a homomorphism τ of T onto T^* such that $\psi^*=\tau\psi$. Thus, we obtain the following

Theorem 2. *Let ψ^* be a homomorphism of left simple semigroup S onto a semigroup T^* which contains at least one idempotent. Then, there exists a homomorphism τ of T onto T^* such that $\psi^*=\tau\psi$.*

References

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