124. On Singular Perturbation of Linear Partial Differential Equations with Constant Coefficients. II

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§0. Introduction. Professor M. Nagumo proved in his recent note[1] the following theorem on the stability of linear partial differential equations of the form

(0) $$L_\varepsilon(u)=\sum_{\mu=0}^{l} P_\mu(\partial_x, \varepsilon)\partial_t^\mu u=f_\varepsilon(t, x).$$ [2]

Definition. We say that the equation (0) is *H_p-stable for $\varepsilon\downarrow 0$ in $0\leqq t\leqq T$ with respect to a particular solution $u=u_0(t)$ of (0) for $\varepsilon=0$,* if $u_\varepsilon(t)\to u_0(t)$ in $H_{p,x}$ uniformly for $0\leqq t\leqq T$, whenever $f_\varepsilon(t, x)\to f_0(t, x)$ in $H_{p,x}$ uniformly for $0\leqq t\leqq T$, and $u_\varepsilon(t)=u(t, x, \varepsilon)$ is a generalized H_p-solution of (0) such that $\partial_t^{j-1}u_\varepsilon(0)\to\partial_t^{j-1}u_0(0)$ in $H_{p,x}$ $(j=1,\cdots, l)$.

Theorem A. *Let degree of $\{P_\mu(\xi, \varepsilon)-P_\mu(\xi, 0)\}\leqq k$ $(\mu=0,\cdots, l)$ and let $u=u_0(t)$ be an l-times continuously $H_{p+k,x}$-differentiable solution of (0) for $\varepsilon=0$ in $0\leqq t\leqq T$. In order that (0) be H_p-stable for $\varepsilon\downarrow 0$ with respect to $u=u_0(t)$ in $0\leqq t\leqq T$, it is necessary and sufficient that there exist constants $\varepsilon_0>0$ and C such that:*

$$\sup_{\xi\in E^m} Y_j(t, \xi, \varepsilon)\leqq C \quad \text{for } 0\leqq t\leqq T,\ 0<\varepsilon\leqq\varepsilon_0$$

and

$$\sup_{\xi\in E^m}\int_0^T |P_l(\xi, \varepsilon)^{-1}Y_l(t, \xi, \varepsilon)|\,dt\leqq C \quad \text{for } 0<\varepsilon\leqq\varepsilon_0$$

where $Y=Y_j(t, \xi, \varepsilon)$ are matricial solutions of

$$\sum_{\mu=0}^{l} P_\mu(i\xi, \varepsilon)(d/dt)^\mu y=0$$

with the initial conditions $\partial_t^{k-1}Y_j(0, \xi, \varepsilon)=\delta_{jk}1$ $(k=1,\cdots, l)$.

In this note we are concerned with the H_p-stability of the equation

$$\varepsilon\cdot\partial_t^2 u+a\cdot\partial_t u+Q(\partial_x)u=f_\varepsilon(t, x)$$

where a is a complex constant and $Q(i\xi)$ is a polynomial in $\xi\in E^m$, and making use of Theorem A we decide the structure of $Q(i\xi)$ in order that this equation be H_p-stable.[3]

I want to take this opportunity to thank Professor M. Nagumo and Mr. K. Ise for their constant assistance.

§1. Main theorems. In this section we shall exhibit three theorems on H_p-stability of the equation

(1.1) $$\varepsilon\cdot\partial_t^2 u+a\cdot\partial_t u+Q(\partial_x)u=f_\varepsilon(t, x).$$

The fundamental solutions of the equation

$$\varepsilon(d^2/dt^2)y+a(d/dt)y+Q(i\xi)y=0$$

are represented by

1) M. Nagumo: On singular perturbation of linear partial differential equations with constant coefficients. I, Proc. Japan Acad., **35**, 449 (1959).

2) We use the same notations and terminology with Nagumo 1).

3) In this note we say H_p-stable for simplicity.

(1.2) $Y_1(t,\xi,\varepsilon)\equiv Y_1[t,\lambda_1,\lambda_2]=1/(\lambda_2-\lambda_1)\{\lambda_2\exp(\lambda_1 t)-\lambda_1\exp(\lambda_2 t)\}$,

(1.3) $Y_2(t,\xi,\varepsilon)\equiv Y_2[t,\lambda_1,\lambda_2]=1/(\lambda_2-\lambda_1)\{\exp(\lambda_2 t)-\exp(\lambda_1 t)\}$,

where $\lambda_1=\lambda_1(\xi,\varepsilon)=1/2\varepsilon\{-a-\sqrt{a^2-4\varepsilon Q(i\xi)}\}$[4]

and $\lambda_2=\lambda_2(\xi,\varepsilon)=1/2\varepsilon\{-a+\sqrt{a^2-4\varepsilon Q(i\xi)}\}$.

Applying Theorem A to the equation (1.1) we obtain the next

Theorem A′. *The equation* (1.1) *is H_p-stable, if and only if*

$$(1.4)\quad\begin{cases}(\mathrm{I})\ \ \sup\limits_{\xi\in E^m}|Y_1(t,\xi,\varepsilon)|\leqq C & \textit{for } 0\leqq t\leqq T,\ 0<\varepsilon\leqq\varepsilon_0,\\ (\mathrm{II})\ \ \sup\limits_{\xi\in E^m}|Y_2(t,\xi,\varepsilon)|\leqq C & \textit{for } 0\leqq t\leqq T,\ 0<\varepsilon\leqq\varepsilon_0,\\ (\mathrm{III})\ \ \sup\limits_{\xi\in E^m}\displaystyle\int_0^T\left|\frac{1}{\varepsilon}\cdot Y_2(t,\xi,\varepsilon)\right|dt\leqq C & \textit{for } 0<\varepsilon\leqq\varepsilon_0.\end{cases}$$

Making use of these results we shall obtain the following theorems.

Theorem 1. *If the equation* (1.1) *is H_p-stable, then the constant a does not vanish and* $\mathrm{Re}\,a$[5] *is non-negative.*

Theorem 2. *Let* $\mathrm{Re}\,a>0$. *Then, in order that the equation* (1.1) *be H_p-stable, it is necessary and sufficient that there exist constants C and R such that*

$$(1.5)\quad\begin{cases}(\mathrm{I})\ \ Q_1(\xi)+C>0 & \textit{for all } \xi\in E^m\\ (\mathrm{II})\ \ Q_2^2(\xi)\leqq R(Q_1(\xi)+C) & \textit{for all } \xi\in E^m\end{cases}$$

where $Q_1(\xi)=\mathrm{Re}\,Q(i\xi)$ *and* $Q_2(\xi)=\mathrm{Im}\,Q(i\xi)$.

Theorem 3. *Let* $\mathrm{Re}\,a=0$ *and* $\mathrm{Im}\,a\neq 0$. *Then, in order that the equation* (1.1) *be H_p-stable, it is necessary and sufficient that there exist constants C and K such that*

$$(1.6)\quad\begin{cases}(\mathrm{I})\ \ Q_2(\xi)=K; & \textit{for all } \xi\in E^m\\ (\mathrm{II})\ \ Q_1(\xi)\geqq C; & \textit{for all } \xi\in E^m.\end{cases}$$

Proof of Theorem 1. i) First we assume $a=0$. We put $\sqrt{-Q(i\xi_0)}=\alpha+\beta i$, $\alpha\geqq 0$, with a fixed $\xi_0\in E^m$. Then, for any fixed $t>0$, if $\alpha>0$,

$$(1.7)\quad\begin{aligned}|Y_2(t,\xi_0,\varepsilon)|&=(1/|\lambda_2-\lambda_1|)|\{\exp(\lambda_2 t)-\exp(\lambda_1 t)\}|\\&\geqq(\sqrt{\varepsilon}/2|\alpha+\beta i|)\{\exp((\alpha/\sqrt{\varepsilon})t)-\exp(-(\alpha/\sqrt{\varepsilon})t)\}\to\infty \quad\text{as } \varepsilon\downarrow 0,\end{aligned}$$

and, if $\alpha=0$, $\beta\neq 0$, or if $\alpha=\beta=0$, we have the next, respectively,

$$(1.8)\quad\int_0^T\left|\frac{1}{\varepsilon}\cdot Y_2(t,\xi_0,\varepsilon)\right|dt=\frac{1}{|\beta|\sqrt{\varepsilon}}\int_0^T\left|\sin\frac{\beta t}{\sqrt{\varepsilon}}\right|dt\to\infty \quad\text{as } \varepsilon\downarrow 0,$$

$$(1.9)\quad\int_0^T\left|\frac{1}{\varepsilon}\cdot Y_2(t,\xi_0,\varepsilon)\right|dt=\int_0^T\frac{t}{\varepsilon}\,dt\to\infty \quad\text{as } \varepsilon\downarrow 0.$$

Applying (1.7), (1.8) or (1.9) to Theorem A′, we see that for $a=0$ the equation (1.1) can not be H_p-stable.

ii) Now we assume $\mathrm{Re}\,a<0$. Since for a fixed ξ_0,

$$\lambda_1(\xi_0,\varepsilon)=(1/2\varepsilon)\{-a-\sqrt{a^2-4\varepsilon Q(i\xi_0)}\}=o(\varepsilon)/\varepsilon,$$

$$\lambda_2(\xi_0,\varepsilon)=(1/2\varepsilon)\{-a+\sqrt{a^2-4\varepsilon Q(i\xi_0)}\}=(1/\varepsilon)(-a+o(\varepsilon)),$$

4) For a complex number b, $\sqrt{b}$ denotes a square-root of b whose real part is non-negative.

5) For a complex number b, $\mathrm{Re}\,b$ denotes real part of b, and $\mathrm{Im}\,b$ denotes imaginary part.

and $|\lambda_2-\lambda_1|=(1/\varepsilon)|a+o(\varepsilon)|$, using the assumption $\operatorname{Re} a<0$, we get for sufficiently small $\varepsilon>0$,

$$\operatorname{Re}\lambda_1(\xi_0,\varepsilon)\leqq -(1/4\varepsilon)\operatorname{Re} a,\quad \operatorname{Re}\lambda_2(\xi_0,\varepsilon)\geqq -(1/2\varepsilon)\operatorname{Re} a,$$

and
$$|\lambda_2-\lambda_1|\leqq(2/\varepsilon)|a|.$$

Hence for any fixed $t>0$,

$$|Y_2(t,\xi,\varepsilon)|\geqq\frac{\varepsilon}{2|a|}\left\{\exp\left(-\frac{1}{2\varepsilon}\operatorname{Re}(at)\right)-\exp\left(-\frac{1}{4\varepsilon}\operatorname{Re}(at)\right)\right\}\to\infty \quad \text{as } \varepsilon\downarrow 0,$$

and consequently from Theorem A′ (1.1) can not be H_p-stable.

§2. Lemmas for the proofs of Theorems 2 and 3.

Lemma 1. Let Z be a complex number. Then,

$$\{\operatorname{Re}(\pm\sqrt{Z})\}^2=\frac{1}{2}\left\{\operatorname{Re} Z+\sqrt{(\operatorname{Re} Z)^2+(\operatorname{Im} Z)^2}\right\}$$
$$=\frac{1}{2}\cdot\frac{(\operatorname{Im} Z)^2}{(-\operatorname{Re} Z)+\sqrt{(\operatorname{Re} Z)^2+(\operatorname{Im} Z)^2}}.$$

Especially we have the following equalities which will be used several times. Let a and Q be complex numbers. Then,

(2.1)
$$\{\operatorname{Re}(\pm\sqrt{a^2-4\varepsilon Q})\}^2=\frac{1}{2}\{(a_1^2-a_2^2-4\varepsilon Q_1)+\sqrt{(a_1^2-a_2^2-4\varepsilon Q_1)^2+(2a_1a_2-4\varepsilon Q_2)^2}\}$$

and

(2.2)
$$\{\operatorname{Re}(\pm\sqrt{a^2-4\varepsilon Q})\}^2=\frac{1}{2}\cdot\frac{(2a_1a_2-4\varepsilon Q_2)^2}{(4\varepsilon Q_1+a_2^2-a_1^2)+\sqrt{(a_1^2-a_2^2-4\varepsilon Q_1)^2+(2a_1a_2-4\varepsilon Q_2)^2}}$$

where $a_1=\operatorname{Re} a$, $a_2=\operatorname{Im} a$, $Q_1=\operatorname{Re} Q$, and $Q_2=\operatorname{Im} Q$.

Proof is omitted.

Lemma 2. We put $\lambda(Q,\varepsilon)=(1/2\varepsilon)\{-a\pm\sqrt{a^2-4\varepsilon Q}\}$ with $a_1>0$. Then, for any $R>0$ and $\varepsilon_0>0$ there exists a constant C such that

$$\operatorname{Re}\lambda(Q,\varepsilon)\leqq C \quad \text{for } |Q|=\sqrt{Q_1^2+Q_2^2}\leqq R,\quad 0<\varepsilon\leqq\varepsilon_0$$

where notations are the same with Lemma 1.

Proof. Using $|Q|\leqq R$ and $0<\varepsilon\leqq\varepsilon_0$, it follows from (2.1) that with large positive constants C_1, C_2, and C_3,

$$\{\operatorname{Re}(\pm\sqrt{a^2-4\varepsilon Q})\}^2\leqq(1/2)\{(a_1^2-a_2^2-4\varepsilon Q_1)+(a_1^2+a_2^2)+C_2\cdot\varepsilon\}\leqq a_1^2+C_3\cdot\varepsilon.$$

As $a_1>0$, we get then

$$\operatorname{Re}\lambda(Q,\varepsilon)=(1/2\varepsilon)\{-a_1+\operatorname{Re}(\pm\sqrt{a^2-4\varepsilon Q})\}\leqq(1/2)\cdot C_3.$$

Lemma 3. As another representation of (1.2), we have

$$Y_1[t,\lambda_1,\lambda_2]=\int_0^1\{\exp(\lambda_1 t)-(\lambda_1 t)\exp(\lambda_1+\theta(\lambda_2-\lambda_1))t\}d\theta. \tag{2.3}$$

Proof. Put $F(z)=Z\exp(\lambda_1 t)-\lambda_1\exp(Zt)$. Then,

$$Y_1[t,\lambda_1,\lambda_2]=\int_0^1\left(\frac{d}{dz}F\right)_{z=\lambda_1+\theta(\lambda_2-\lambda_1)}d\theta=\int_0^1\{\exp(\lambda_1 t)-(\lambda_1 t)\exp(Zt)\}_{z=\lambda_1+\theta(\lambda_2-\lambda_1)}d\theta.$$

Lemma 4. Put $\lambda=\dfrac{1}{2\varepsilon}\{-a\pm\sqrt{a^2-4\varepsilon Q}\}$.

If there exist constants $\varepsilon_0>0$, $\delta>0$, and R such that for any $\varepsilon(0<\varepsilon\leqq\varepsilon_0)$, $\operatorname{Re}\lambda\leqq R$ and $|\sqrt{a^2-4\varepsilon Q}|\geqq\delta>0$, then $|Y_1[t,\lambda_1,\lambda_2]|$ is bounded.

Proof. Since $\left|\frac{\lambda}{\lambda_2-\lambda_1}\right|\leqq\frac{1}{2}\left\{\left|\frac{a}{\sqrt{a^2-4\varepsilon Q}}\right|+1\right\}$, and $|\exp(\lambda t)|=\exp(\mathrm{Re}\cdot(\lambda t))$, we can easily prove this lemma.

Lemma 5. Let $a\neq 0$ and $\varepsilon_0>0$. If $\mathrm{Re}\,\lambda_\nu$ ($\nu=1,2$) are bounded above for $0<\varepsilon\leqq\varepsilon_0$, then, (I) in (1.4) implies (II) and (III) in (1.4).

Proof. It is easy to see that

$$Y_1[t,\lambda_1,\lambda_2]=\frac{1}{2}\left\{\exp(\lambda_1 t)+\exp(\lambda_2 t)+\frac{a}{\varepsilon}\cdot Y_2[t,\lambda_1,\lambda_2]\right\}.$$

Then, for $a\neq 0$, $\left|\frac{1}{\varepsilon}\cdot Y_2[t,\lambda_1,\lambda_2]\right|$ is bounded, and consequently

$$|Y_2[t,\lambda_1,\lambda_2]| \text{ and } \int_0^T\left|\frac{1}{\varepsilon}\cdot Y_2[t,\lambda_1,\lambda_2]\right|dt \text{ are bounded.}$$

§3. The proofs of Theorem 2 and Theorem 3. **Proof of Theorem 2.** *Necessity of the conditions* (1.5). If (I) of (1.5) did not hold, then there would exist a sequence $\{\xi_\nu\}$ such that $Q_1(\xi_\nu)\to-\infty$ as $\nu\to\infty$. We can take a sequence $\{\varepsilon_\nu\}$ ($\varepsilon_\nu>0$) such that

$$\varepsilon_\nu\cdot Q_1(\xi_\nu)\to-\infty \text{ and } \varepsilon_\nu\downarrow 0 \text{ as } \nu\to\infty. \tag{3.1}$$

Then, it follows from (2.1) and (3.1) that

$$\{\mathrm{Re}(\pm\sqrt{a^2-4\varepsilon_\nu Q(i\xi_\nu)})\}^2\geqq(a_1^2-a_2^2-4\varepsilon_\nu Q_1(\xi_\nu))\to\infty \quad\text{as } \nu\to\infty.$$

Then there exists a constant $C>0$ such that for large ν

$$\begin{aligned}\mathrm{Re}\,\lambda_1(\xi_\nu,\varepsilon_\nu)&=\frac{1}{2\varepsilon_\nu}\{-a_1-\mathrm{Re}\sqrt{a^2-4\varepsilon_\nu Q(i\xi_\nu)}\}\leqq 0,\\ \mathrm{Re}\,\lambda_2(\xi_\nu,\varepsilon_\nu)&=\frac{1}{2\varepsilon_\nu}\{-a_1+\mathrm{Re}\sqrt{a^2-4\varepsilon_\nu Q(i\xi_\nu)}\}\geqq\frac{1}{\varepsilon_\nu}\cdot C.\end{aligned}\tag{3.2}$$

And, as $|\sqrt{a^2-4\varepsilon_\nu Q(i\xi_\nu)}|\geqq|\mathrm{Re}\sqrt{a^2-4\varepsilon_\nu Q(i\xi_\nu)}|\to\infty$ for $\nu\to\infty$, we have

$$\left|\frac{\lambda}{\lambda_2-\lambda_1}\right|=\frac{1}{2}(o(\nu)+1).\tag{3.3}$$

Applying (3.2) and (3.3) to (1.2) we get

$$|Y_1(t,\xi_\nu,\varepsilon_\nu)|\to\infty \quad\text{as } \nu\to\infty \quad\text{for any fixed } t>0.$$

Then condition (I) of (1.5) is thus necessary.

Now assume that (I) of (1.5) holds, but that (II) of (1.5) did not hold, then there would exist a sequence such that

$$Q_2^2(\xi_\nu)\geqq\nu\cdot|Q_1(\xi_\nu)| \text{ and } Q_2(\xi_\nu)\geqq\nu \text{ for any } \nu.\tag{3.4}$$

Then, for a fixed ε', $0<\varepsilon'<1/4$ and large ν, from (2.2) and (3.4)

$$\{\mathrm{Re}(\pm\sqrt{a^2-4\varepsilon' Q(i\xi_\nu)})\}^2\geqq\frac{C\cdot Q_2(\xi_\nu)^2}{|Q_1(\xi_\nu)|+Q_2(\xi_\nu)}\geqq\frac{1}{2}\cdot C\nu,$$

with a constant $C>0$, hence $|\sqrt{a^2-4\varepsilon' Q(i\xi_\nu)}|\geqq|\mathrm{Re}\sqrt{a^2-4\varepsilon' Q(i\xi_\nu)}|\geqq\sqrt{1/2\cdot C\nu}$. Thus we obtain:

$$\begin{cases}\mathrm{Re}\,\lambda_1(\xi_\nu,\varepsilon')=\frac{1}{2\varepsilon'}\{-a_1-\mathrm{Re}\sqrt{a^2-4\varepsilon' Q(i\xi_\nu)}\}\leqq 0 & \text{for large } \nu,\\ \mathrm{Re}\,\lambda_2(\xi_\nu,\varepsilon')=\frac{1}{2\varepsilon'}\{-a_1+\mathrm{Re}\sqrt{a^2-4\varepsilon' Q(i\xi_\nu)}\}\to\infty & \text{as } \nu\to\infty\end{cases}\tag{3.5}$$

and

(3.6) $$\left|\frac{\lambda}{\lambda_2-\lambda_1}\right|=\frac{1}{2}(o(\nu)+1).$$

Applying (3.5) and (3.6) to (1.2), it follows that

$$|Y_1(t,\xi_\nu,\varepsilon')|\to\infty \quad \text{as } \nu\to\infty \quad \text{for fixed } t>0.$$

Thus, the condition (II) of (1.5) is necessary.

Sufficiency of the conditions (1.5). First we shall prove that $\operatorname{Re}\lambda$ are bounded above.

Set $\Xi_a=\{\xi; Q_1(\xi)\leqq a\}$, and take $C'\geqq C$, then from (1.5),

(3.7) $$Q_2^2(\xi)\leqq R\cdot(Q_1(\xi)+C)\leqq 2R\cdot C' \quad \text{for } \xi\in\Xi_{C'},$$

(3.8) $$Q_2^2(\xi)\leqq R\cdot(Q_1(\xi)+C)\leqq 2R\cdot Q_1(\xi) \quad \text{for } \xi\in E^m-\Xi_{C'}.$$

Since on $\Xi_{C'}$, $Q_2^2(\xi)$ is bounded by (3.7), we get from Lemma 2 that $\operatorname{Re}\lambda$ are bounded above.

If $\xi\in E^m-\Xi_{C'}$, using (3.8) we get by (2.1) that, for sufficiently large C' and small $\varepsilon_0>0$,

$$\{\operatorname{Re}(\pm\sqrt{a^2-4\varepsilon Q})\}^2\leqq\frac{1}{2}\{(a_1^2-a_2^2-4\varepsilon Q_1)+(a_1^2+a_2^2+4\varepsilon Q_1)\}=a_1^2 \quad \text{for } 0<\varepsilon\leqq\varepsilon_0,$$

hence $$\operatorname{Re}\lambda=\frac{1}{2\varepsilon}\{-a_1+\operatorname{Re}(\pm\sqrt{a^2-4\varepsilon Q})\}\leqq\frac{1}{2}\{-a_1+a_1\}=0.$$

Thus, $\operatorname{Re}\lambda(\xi,\varepsilon)$ are bounded above on $\xi\in E^m$, $0<\varepsilon\leqq\varepsilon_0$. Now by Lemma 5 and Theorem A′ we have only to prove the boundedness of $|Y_1(t,\xi,\varepsilon)|$. We put

(3.9) $$H_\varepsilon=\{\xi; |4\varepsilon Q_1(\xi)+a_2^2-a_1^2|\geqq\frac{1}{4}a_1^2\}.$$

If $\xi\in H_\varepsilon$, then $|\sqrt{a^2-4\varepsilon\cdot Q}|\geqq\sqrt{|a_1^2-a_2^2-4\varepsilon Q_1|}\geqq\frac{1}{2}a_1$, thus by Lemma 4 $|Y_1(t,\xi,\varepsilon)|$ is bounded on H_ε.

If $\xi\in E^m-H_\varepsilon$ and $a_2\neq 0$, we have $|4\varepsilon Q_1|\leqq\frac{1}{4}a_1^2+|a_2^2-a_1^2|$ from (3.9). Thus from (3.7) and (3.8)

(3.10) $$\begin{aligned}|4\varepsilon Q_2|&\leqq 4\varepsilon\operatorname{Max}\{\sqrt{2RC'},\ \sqrt{2R|Q_1|}\}\\ &\leqq 4\varepsilon\operatorname{Max}\{\sqrt{2RC'},\ \sqrt{1/2\cdot R\cdot\varepsilon^{-1}(1/4\cdot a_1^2+|a_2^2-a_1^2|)}\}.\end{aligned}$$

Hence, for sufficiently small $\varepsilon_0>0$

$$|\sqrt{a^2-4\varepsilon Q}|\geqq\sqrt{|2a_1a_2-4\varepsilon Q_2|}\geqq\sqrt{|a_1a_2|}>0 \quad (0<\varepsilon\leqq\varepsilon_0),$$

and consequently by Lemma 4 $|Y_1(t,\xi,\varepsilon)|$ is bounded.

If $a_2=0$, by (3.10), there exists $\varepsilon_0>0$ such that

(3.11) $$(4\varepsilon Q_2)^2\leqq\frac{1}{2}\cdot a_1^4 \quad \text{for } \xi\in E^m-H_\varepsilon,\quad 0<\varepsilon\leqq\varepsilon_0,$$

and as $|4\varepsilon Q_1-a_1^2|\leqq\frac{1}{4}\cdot a_1^2$ for $\xi\in E^m-H_\varepsilon$, we get from (2.1) and (3.11)

$$\{\operatorname{Re}(\pm\sqrt{a^2-4\varepsilon Q})\}^2\leqq\frac{1}{2}\left\{\frac{1}{4}\cdot a_1^2+\sqrt{\frac{1}{16}\cdot a_1^4+\frac{1}{2}\cdot a_1^4}\right\}=\frac{1}{2}\cdot a_1^2, \text{ and consequently}$$

$$\operatorname{Re}\lambda=\frac{1}{2\varepsilon}\{-a_1+\operatorname{Re}(\pm\sqrt{a^2-4\varepsilon Q})\}\leqq\frac{1}{2\varepsilon}\left\{-a_1\pm\frac{1}{\sqrt{2}}a_1\right\}\leqq-\frac{1}{8\varepsilon}\cdot a_1.$$

Hence $$\operatorname{Re}\{\lambda_1+\theta(\lambda_2-\lambda_1)\}\leqq-\frac{1}{8\varepsilon}\cdot a_1 \quad \text{for } 0\leqq\theta\leqq 1,\ \xi\in E^m-H_\varepsilon.$$

On the other hand, from (3.9) and (3.11) we have for $\xi\in E^m-H_\varepsilon$

$$|\lambda_1|\leqq\frac{1}{2\varepsilon}\{a_1+|\sqrt{a^2-4\varepsilon Q}\,|\}\leqq\frac{1}{2\varepsilon}\{a_1+\sqrt{|a_1^2-4\varepsilon Q_1|+|4\varepsilon Q_2|}\}\leqq\frac{1}{\varepsilon}\cdot a_1.$$

Then by (2.3), we get for $t>0$

$$|Y_1(t,\xi,\varepsilon)|\leqq\Big(1+\frac{1}{\varepsilon}a_1t\Big)\exp\Big(-\frac{1}{8\varepsilon}a_1t\Big), \tag{3.12}$$

hence $|Y_1(t,\xi,\varepsilon)|$ is bounded. Q.E.Q.

Proof of Theorem 3. *Necessity of the conditions* (1.6). If (I) of (1.6) did not hold, then we can take a sequence such that with some constant $C>0$

$$|Q_2(\xi_\nu)|\geqq C|\xi_\nu|. \tag{3.13}$$

We can take a sequence $\{\varepsilon_\nu\}$, $\varepsilon_\nu>0$, such that

$$4\varepsilon_\nu\{|Q_1(\xi_\nu)|+|Q_2(\xi_\nu)|\}\leqq\frac{1}{2}a_2^2 \quad\text{and}\quad \varepsilon_\nu\to 0 \quad\text{as } \nu\to\infty$$

then by (2.2)

$$\{\mathrm{Re}\,(\pm\sqrt{a^2-4\varepsilon_\nu Q(i\xi_\nu)})\}^2\geqq\frac{\varepsilon_\nu^2Q_2^2(\xi_\nu)}{a_2^2}.$$

Hence

$$\begin{cases}\mathrm{Re}\,\lambda_1(\xi_\nu,\varepsilon_\nu)\leqq 0,\\ \mathrm{Re}\,\lambda_2(\xi_\nu,\varepsilon_\nu)\geqq\dfrac{|Q_2(\xi_\nu)|}{2|a_2|}\geqq\dfrac{C|\xi_\nu|}{2|a_2|},\end{cases} \tag{3.14}$$

and

$$|\lambda_2-\lambda_1|=\frac{1}{\varepsilon_\nu}|\sqrt{a^2-4\varepsilon_\nu Q(\xi_\nu)}\,|\leqq\frac{1}{\varepsilon_\nu}\sqrt{|a_2^2|+\Big|\frac{a_2^2}{2}\Big|}<\frac{2|a_2|}{\varepsilon_\nu}. \tag{3.15}$$

Using (3.14) and (3.15), we get

$$\int_0^T\Big|\frac{1}{\varepsilon_\nu}\cdot Y_2(t,\xi_\nu,\varepsilon_\nu)\Big|dt\geqq\frac{1}{2|a_2|}\int_0^T\Big\{\exp\Big(\frac{C|\xi_\nu|}{2|a_2|}t\Big)-1\Big\}dt\to\infty \quad\text{as } \nu\to\infty.$$

Hence from Theorem A$'$ (1.1) can not be H_p-stable.

The proof of the necessity of (II) in (1.6) goes in the same way as that of (I) of (1.5) in Theorem 2.

Sufficiency of the conditions (1.6). We take $\varepsilon_0>0$ such that $a_2^2-4\varepsilon_0|c|\geqq(1/2)a_2^2$, then from (1.6),

$$a_2^2+4\varepsilon\cdot Q_1\geqq\frac{1}{2}\cdot a_2^2 \quad\text{for } 0<\varepsilon\leqq\varepsilon_0. \tag{3.16}$$

Therefore, by (2.2)

$$\{\mathrm{Re}\,(\pm\sqrt{a^2-4\varepsilon Q})\}^2\leqq\frac{(4\varepsilon Q_2)^2}{2a_2^2}=\frac{8K^2}{a_2^2}\varepsilon^2, \tag{3.17}$$

and consequently

$$\mathrm{Re}\,\lambda=\frac{1}{2\varepsilon}\cdot\mathrm{Re}(\pm\sqrt{a^2-4\varepsilon Q})\leqq\frac{\sqrt{2}\,|K|}{|a_2|}, \tag{3.18}$$

and by (3.16)

$$|\sqrt{a^2-4\varepsilon Q}\,|\geqq\sqrt{|a_2^2+4\varepsilon Q_1|}\geqq\frac{1}{\sqrt{2}}|a_2|. \tag{3.19}$$

Applying (3.18) and (3.19) to Lemma 4, it follows that $|Y_1(t,\xi,\varepsilon)|$ is bounded, so by Lemma 5 and Theorem A$'$, the equation (1.1) is H_p-stable. Q.E.D.