134. *Representation of Some Topological Algebras. III*

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7. On the condition (ii). It is easy to see that a semi-simple algebra satisfies the condition $(*)$[1] but not the condition (ii)[1] in general. Let E be an algebra, and let $u\in E$; then we denote by $(u)_r$ the right ideal generated by u, that is, the set of all elements $\lambda u+ux$, where λ runs over the scalar field and x over the whole E; we write $(u)_l$ the left ideal generated by u.

LEMMA 1. *For a semi-simple algebra E, each one of the following conditions is equivalent to the condition* (ii):

(1) *For any two non-zero elements $u, v\in E$, we have $uE\frown Ev \neq \{0\}$.*

(2) *For any two non-zero elements $u, v\in E$, we have $(u)_r\frown(v)_l \neq \{0\}$.*

Proof. It is clear that the condition (ii) implies (1) and (1) implies (2). To prove the implication (2)$\rightarrow$(ii), let us suppose that an algebra E satisfies the condition (2) and not (ii). Then there exist two non-zero elements $u, v\in E$ such that $uxv=0$ for every $x\in E$. Since E is semi-simple, we can find an element $a\in E$ with $ua\neq 0$, and so by (2), there exists a non-zero element $w=\alpha ua+uab=\beta v+cv\in(ua)_r\frown(v)_l$, where α, β are two numbers and $b, c\in E$. Now, if $w^2=0$, then for any number λ and any $x\in E$, we have

$$\lambda w+xw-\lambda w-xw+\lambda^2w^2+\lambda wxw+\lambda xw^2+xwxw=0,$$

since $wxw=(\alpha ua+uab)x(\beta v+cv)=0$; it follows that w belongs to the radical of E, and so $w=0$, which is a contradiction. Thus $w^2=\alpha\beta uav+\alpha uacv+\beta uabv+uabcv\neq 0$. But this is absurd since $uEv=\{0\}$.

LEMMA 2. *For an algebra E with a minimal left ideal L such that $L^2\neq\{0\}$, each one of the following conditions is equivalent to the condition* (ii):

(1) *For any non-zero element $u\in E$, we have $uE\frown L\neq\{0\}$.*

(2) *For any non-zero element $u\in E$, we have $uL\neq\{0\}$.*

Proof. Since $L^2\neq\{0\}$, we can find an idempotent $p\in E$ such that $L=Ep$. The implication (ii)$\rightarrow$(1) is obvious, because $uEp\subseteq uE\frown L$. If there exists a non-zero element $ux\in uE\frown Ep$, then we have $ux=ap$ for some $a\in E$, and hence $0\neq ap=uxp\in uL$, proving the implication (1)$\rightarrow$(2). Now suppose that the condition (2) is satisfied, and let u, v

1) Cf. S. Kasahara: Representation of some topological algebras. I, II, Proc. Japan Acad., **34**, 355-360 (1958); **35**, 89-94 (1959).

be two non-zero elements of E. Then since $vEp \neq \{0\}$, we can find an element $b \in E$ such that $vbp \neq 0$. We shall now show that $Evbp \neq \{0\}$. Suppose that $Evbp = \{0\}$; then the set I of all elements xp specified by the relation $Exp = \{0\}$ contains a non-zero element vbp, and as can easily be seen, I is a left ideal contained in the minimal left ideal Ep. Hence $I = Ep$. It follows that $p \in I$, and we have $Ep = \{0\}$, which is absurd. Thus $Evbp \neq \{0\}$. Now since the left ideal $Evbp$ is contained in the minimal left ideal Ep, we have $Evbp = Ep$ and so we have $uEvbp = uEp \neq \{0\}$, which implies $uEv \neq \{0\}$. This completes the proof.

8. Continuity of the ring multiplication. LEMMA 1. *Let E be a topological algebra and $p \in E$ be a non-zero idempotent. Then the topology of Ep (resp. pE) induced from E is the finest topology having the property that the linear mapping φ of E into Ep (resp. pE), defined by $\varphi(x) = xp$ (resp. $\varphi(x) = px$), is continuous.*

Proof. It will suffice to give a proof only to Ep. For any neighbourhood U of 0 in E, we can find a neighbourhood V of 0 in E such that $Vp \subseteq U$. Hence $Vp \subseteq U \frown Ep$. Conversely, if $xp \in V \frown Ep$, then $xp = (xp)p \in Vp$, and so we have $V \frown Ep \subseteq Vp$.

Let E and F be two vector spaces constituting a separated dual system. We say that a topology on E is *compatible with the duality* between E and F if, with the topology, E is a topological vector space (not necessarily locally convex) and the dual of E is F.

THEOREM 9′. *Let E be a topological algebra satisfying the condition* (ii), *and let $p, q \in E$ be two non-zero idempotents of rank* 1. *Then:*

(1) *With the induced topology from E, the topological vector spaces Ep (resp. pE) and Eq (resp. qE) are isomorphic.*

(2) *If the topology of Ep (resp. pE) induced from E is compatible with the duality between Ep and pE (resp. pE and Ep), then the topology of Eq (resp. pE) induced from E is compatible with the duality between Eq and qE (resp. qE and Eq).*

Proof. It will suffice to give a proof to Ep. By Lemma 4 of section 4, we can take two elements $a, b \in E$ such that $p = aqb$ and $q = bpa$. As was shown in the proof of Theorem 9, the mapping φ defined by $\varphi(xp) = xpaq$ is an (algebraic) isomorphism of Ep onto Eq, and the mapping $xq \to xqbp$ is nothing more than the inverse mapping of φ. But then the ring multiplication being separately continuous, φ is a homeomorphism.

Let us now turn to prove the assertion (2). Let x' be an arbitrary continuous linear form on Eq. Since the mapping φ defined above is continuous, $x' \circ \varphi$ is a continuous linear form on Ep. Therefore by the assumption, the linear form $x' \circ \varphi$ is represented by an element pz of pE. That is, for any $xp \in Ep$, we have $\langle xp, x' \circ \varphi \rangle p = pzxp$. It follows that, for any $xq \in Eq$,

$$\langle xq, x'\rangle p = \langle xqbp, x'\circ\varphi\rangle p = pzxqbp = \lambda p$$

for some λ. Hence we have $qbpzxq = qbpzxqbpa = \lambda qbpa = \lambda q$, and so $qbpzxq = \langle xq, x'\rangle q$ for any $xq \in Eq$. Thus the dual of Eq is identical with qE, and the proof of (2) is completed.

Let E be an algebra satisfying the condition (ii). We say that a topology τ compatible with the structure of vector space of E is *compatible with a non-zero idempotent of rank* 1 if there exists a non-zero idempotent $p \in E$ of rank 1 such that the topology τ is compatible with the duality between Ep and pE.

Let X be a locally convex Hausdorff vector space, $\mathfrak{S}$ a covering of X consisting of bounded sets in X, and E a subalgebra of $\mathcal{L}(X, X)$ containing all continuous linear mappings of finite rank. Then it is easy to see that the topology of uniform convergence on members of $\mathfrak{S}$ is compatible with a non-zero idempotent of rank 1.

Let X be a vector space. Two topologies τ_1 and τ_2 compatible with the structure of vector space of X, are said to have the *same dual* if the dual of X by τ_1 is identical with that by τ_2.

LEMMA 2. *Let E be an algebra satisfying the condition* (ii), *and $\mathfrak{T}$ the set of all topologies compatible with the structure of vector space of E. Further let $\tau_0 \in \mathfrak{T}$ be compatible with a non-zero idempotent of rank* 1. *Then each topology τ compatible with the structure of algebra of E and satisfying the following condition is compatible with a non-zero idempotent of rank* 1:

There are a finite number of topologies $\tau_1, \tau_2, \cdots, \tau_{n+1}, \tau'_0, \tau'_1, \cdots, \tau'_{n+1}(=\tau) \in \mathfrak{T}$ such that τ_{i+1} is coarser than τ'_i for any $i = 0, 1, \cdots, n$ and that, for each $i = 0, 1, \cdots, n+1$, τ_i and τ'_i have the same dual.

Proof. Let $p \in E$ be a non-zero idempotent of rank 1. Suppose that a topology τ is compatible with the structure of algebra of E and satisfies the condition. Then since the ring multiplication in E is separately continuous for the topology τ, it is sufficient to show that every linear form x' on Ep continuous for the topology τ can be represented by an element of pE. Now put $\langle x, x'_1\rangle = \langle xp, x'\rangle$, then we obtain a linear form x'_1 on E continuous for the topology τ. Hence by the condition, x'_1 is continuous for the topology τ_{n+1}, and so is for the topology τ'_n. Then again by the condition, x'_1 is continuous for the topology τ_n, and so on. Thus we can conclude that x'_1 is continuous for the topology τ_0, and so is its restriction x' to Ep. This completes the proof.

The following theorem is a simple generalization of a theorem due to Rickart.[2]

THEOREM 11. *Let E be an algebra satisfying the conditions* (i)

2) C. E. Rickart: The uniqueness of norm problem in Banach algebras, Ann. Math., **51**, 615–628 (1950).

and (ii). *Then there exists at most one topology which makes E into a metrizable complete topological algebra.*

Proof. Let τ_1 and τ_2 be two topologies which make E into a metrizable complete topological algebra. We show that the identity mapping of E with τ_1 into E with τ_2 is closed, then in view of the closed graph theorem, we have $\tau_1=\tau_2$. Let $\{x_n\}$ be a sequence in E which converges to 0 for τ_1 and to $a\neq 0$ for τ_2, and let p be a non-zero idempotent of rank 1. By the condition (ii), we have $axp\neq 0$ for some $x\in E$, and so we can find further an element $y\in E$ such that $\lambda p=pyaxp\neq 0$. On the other hand, since the topologies considered are Hausdorff ones, there exist two neighbourhoods U_1, U_2 of 0 in E for the topologies τ_1 and τ_2 respectively such that $\lambda p\in U_i$ $(i=1,2)$ if and only if $|\lambda|\leqq 1$. Now let $\varepsilon>0$ be arbitrary. We can find then two neighbourhoods V_1 and V_2 of 0 in E for the topologies τ_1 and τ_2 respectively such that $pyV_1xp\subseteq \varepsilon U_1$ and $pyV_2xp\subseteq\varepsilon U_2$. From the assumption on the sequence $\{x_n\}$ it follows that $a-x_n\in V_2$ and $x_n\in V_1$ for a sufficiently large positive integer n. Therefore if we put $py(a-x_n)xp=\alpha p$ and $pyx_nxp=\beta p$, we have $|\alpha|\leqq\varepsilon$ and $|\beta|\leqq\varepsilon$, and hence we can conclude $|\lambda|\leqq 2\varepsilon$ since $\lambda p=pyaxp=py\cdot(a-x_n)xp+pyx_nxp$. We have thus reached a contradiction $\lambda=0$ or $pyaxp=0$, which completes the proof.

COROLLARY. *Let X be a locally convex Hausdorff vector space. Then there exists at most one topology which makes $\mathcal{L}(X,X)$ into a metrizable complete topological algebra.*

THEOREM 12. *Let E be a locally convex Hausdorff algebra satisfying the condition* (ii). *Suppose that the ring multiplication is continuous and the topology of E is compatible with a non-zero idempotent of rank* 1. *Then there exists a normed vector space X such that E is mapped, by a continuous isomorphism, onto a subalgebra of $\mathcal{L}_b(X,X)$*[3] *containing all continuous linear mappings of finite rank. Consequently, the topology of E can be defined by a family of norms. If in addition the topological algebra E is complete, we can take a Banach space as X.*

Proof. Let U be a neighbourhood of 0 in E such that $\lambda p\in U$ if and only if $|\lambda|\leqq 1$. We can find then, by the assumption, a neighbourhood V of 0 in E such that $VVp\subseteq U$. Now, for any $px\in pE$, there exists a non-zero number λ for which we have $\lambda px\in V$ and so $\lambda pxVp\subseteq U$. It follows therefore that $|\langle Vp, px\rangle|$ is bounded, that is to say, Vp is bounded in Ep, and hence the vector space Ep with the topology induced from E is normable. Let us denote by X the vector space with the norm determined by the set Vp, and let N be a neighbourhood of

3) $\mathcal{L}_b(X,X)$ denotes $\mathcal{L}(X,X)$ with the topology of uniform convergence on each bounded set in X.

0 in E such that $NN\subseteq V$. Then we have, for some number $\lambda\neq 0$, $\lambda Vp\subseteq N$ and so we have $\lambda NVp\subseteq NN\subseteq V$ or $\lambda N\subseteq W(Vp, Vp)$. It follows that the isomorphism $u\rightarrow\tilde{u}$ of E into $\mathcal{L}_b(X, X)$ defined by $\tilde{u}(xp)=uxp$ is continuous. On the other hand, since the topology of E is compatible with the idempotent p, the dual of X is pE, and hence $\tilde{E}=\{\tilde{u}; u\in E\}$ contains all continuous linear mappings of finite rank. If E is complete, then by Lemma 3 of section 4, Ep is also complete. This completes the proof.

COROLLARY 1. *Let E be a metrizable complete locally convex algebra satisfying the condition* (ii). *Suppose that the topology of E is compatible with a non-zero idempotent of rank* 1. *Then there exists a Banach space X such that E is mapped, by a continuous isomorphism, onto a subalgebra of $\mathcal{L}_b(X, X)$ containing all continuous linear mappings of finite rank.*

COROLLARY 2. *Let X be a locally convex Hausdorff vector space, X' its dual, and E be a subalgebra of $\mathcal{L}(X, X)$ containing all continuous linear mappings of finite rank. If there exists a locally convex Hausdorff topology τ, compatible with the structure of algebra of E and also compatible with a non-zero idempotent of rank* 1, *for which the ring multiplication in E is continuous, then the Mackey topology $\tau(X, X')$ is normable.*

In fact, let $x'\otimes x$ ($x\in X$ and $x'\in X'$) be a non-zero idempotent of rank 1, then on the space $E\circ x'\otimes x=x'\otimes X$ the topology τ is normable, and so identical with the Mackey topology since τ is compatible with the non-zero idempotent of rank 1.

COROLLARY 3. *Let X be a locally convex Hausdorff vector space. Suppose that the topology of X is that of Mackey. Then metrizable complete locally convex topology compatible with the structure of algebra of $\mathcal{L}(X, X)$ and also compatible with a non-zero idempotent of rank* 1 *does not exist except for a topology by a norm. That is, if the algebra $\mathcal{L}(X, X)$ is metrizable complete for a locally convex topology τ compatible with a non-zero idempotent of rank* 1, *then X and the topology τ are normable; more precisely, the topology τ is identical with the topology by the operator norm.*

In fact, by Corollary 2, the space X is normable and complete, and so the conclusion follows from Theorem 11.

COROLLARY 4. *Let X be a locally convex Hausdorff vector space, and E be a subalgebra of $\mathcal{L}(X, X)$ containing all continuous linear mappings of finite rank. If there exists a locally convex Hausdorff topology compatible with the structure of algebra of E for which the ring multiplication is continuous and which induces into X a coarser topology than the original one of X, then X is normable.*

The following corollary is a generalization of a well-known result.[4)]

COROLLARY 5. *Let X be a locally convex Hausdorff vector space, and $\mathfrak{S}$ a covering of X consisting of bounded sets in X. Further let E be a topological subalgebra of $\mathcal{L}_{\mathfrak{S}}(X, X)$ containing all continuous linear mappings of finite rank. If, for a locally convex Hausdorff topology τ coarser than one of the topologies compatible with the duality between E and its dual E', the ring multiplication in E is continuous, then the topology τ is normable. If in addition the topology τ is coarser than the topology of uniform convergence on members of $\mathfrak{S}$, then the space X is also normable.*

Proof. As pointed out before, the topology of uniform convergence on members of $\mathfrak{S}$ is compatible with a non-zero idempotent of rank 1, and hence by Lemma 2, the topology τ is also compatible with a non-zero idempotent of rank 1. Therefore by Corollary 2 above, the Mackey topology $\tau(X, X')$ can be given by a norm, and so by this topology we can define a norm topology on E, which is finer than the topology of uniform convergence on members of $\mathfrak{S}$ and coarser than the topology τ. On the other hand, by the assumption, there exists a topology τ_1 finer than τ and compatible with the duality between E and E'. Consequently the norm topology is also compatible with the duality between E and E', and so the topology τ is identical with the norm topology.

In view of Corollary 2 of Lemma 4 of section 4 and Lemma 2 of section 6, we see that a simple algebra E containing a non-zero idempotent of rank 1 is isomorphic with the algebra of all continuous linear operators of finite rank on a locally convex Hausdorff vector space. Therefore a simple algebra containing a non-zero idempotent of rank 1 and an identity element is of finite dimension.

On the other hand, we have the following

COROLLARY 6. *Let E be a locally convex Hausdorff topological simple algebra. If the ring multiplication in E is continuous and the topology of E is compatible with a non-zero idempotent of rank 1, then E is mapped by a continuous isomorphism into the algebra consisting of all completely continuous linear operators on a normed space.*

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4) See for example S. Kasahara: Quelques conditions pour la normabilité d'un espace localement convexe, Proc. Japan Acad., **32**, 574–578 (1956).