133. On Quasi-normed Spaces. II

By Tomoko KONDA

(Comm. by K. KUNUGI, M.J.A., Dec. 12, 1959)

In this paper, we shall consider some theorems in (QN)-spaces. For definitions and notations, see my paper [2], M. Pavel [3] and S. Rolewicz [4].

First of all, we shall prove the following

Lemma. *If L is a proper subspace of the (QN)-space E with the power r, then for any $\varepsilon>0$ and the element y of E such that $\|y\|=1$, every element x of L satisfies the inequality $\|x-y\|>1-\varepsilon$.*

Proof. We take an element $y_0\in E$ such that $y_0\notin L$ and put $d=\inf_{x\in L}\|y_0-x\|$. Then we have $d>0$. For any $\eta>0$, we select also an element $x_0\in L$ such that $d\leq\|y_0-x_0\|<d+\eta$. The element $y=\dfrac{y_0-x_0}{\|y_0-x_0\|^{1/r}}$ is not contained in L, for if y is in L then y_0 must be in L. Moreover $\|y\|=1$ and for any $x\in L$, $x'=x_0+\|y_0-x_0\|^{1/r}x$ and $x'\in L$, we have

$$\|y-x\|=\left\|\frac{y_0-x_0}{\|y_0-x_0\|^{1/r}}-x\right\|=\frac{1}{\|y_0-x_0\|}\|y_0-x'\|$$
$$>\frac{1}{d+\eta}\|y_0-x'\|\geq\frac{d}{d+\eta}=1-\frac{\eta}{d+\eta}.$$

Since η is arbitrary, we can take η such that $\dfrac{\eta}{d+\eta}<\varepsilon$ and $\eta>0$. Thus we have the desired result.

Theorem I. *A subspace L of a (QN)-space E with the power r is a finite dimensional space if and only if any bounded subset of L is compact.* (For Banach space, see [1, pp. 76-78].)

Proof. Necessary. Let L be n-dimensional. Any element $x\in L$ is of form $x=\lambda_1x_1+\lambda_2x_2+\cdots+\lambda_nx_n$ with a base $\{x_i\}$ of L for $i=1,2,\cdots,n$.

Let $\{y_k\}$ be a bounded sequence in L, then we can write $y_k=\lambda_1^{(k)}x_1+\cdots+\lambda_n^{(k)}x_n$ for $k=1,2,\cdots$. By the boundness of $\{y_k\}$ there exists M such that $\|y_k\|\leq M$ for $k=1,2,\cdots$ and it may be proved that the sum $|\lambda_1^{(k)}|^r+\cdots+|\lambda_n^{(k)}|^r$ is bounded. For if the sum is not bounded, then there exists a sequence of indexes $K_1,K_2,\cdots$ such that

$$|\lambda_1^{(k_m)}|^r+|\lambda_2^{(k_m)}|^r+\cdots+|\lambda_n^{(k_m)}|^r=c_m\geq m.$$

Let $y^*_{k_m}=\dfrac{1}{c_m^{1/r}}y_{k_m}$, then we have

$$\|y^*_{k_m}\|=\frac{1}{c_m}\|y_{k_m}\|\leq\frac{1}{c_m}M\leq\frac{M}{m}$$

and $y^*_{k_m}\to 0$ as $m\to\infty$.

On the other hand, $y^*_{k_m}=\mu_1^{(k_m)}x_1+\mu_2^{(k_m)}x_2+\cdots+\mu_n^{(k_m)}x_n$ where

$$\mu_i^{(k_m)}=\frac{1}{c_m^{1/r}}\lambda_i^{(k_m)}.$$

Thus we have

$$|\mu_1^{(k_m)}|^r+\cdots+|\mu_n^{(k_m)}|^r=1.$$

This means that the sequence $\{\mu_i^{(k_1)},\mu_i^{(k_2)},\cdots\}$ is bounded and converges to $\mu_i^{(0)}$. It is clear that $|\mu_1^{(0)}|+|\mu_2^{(0)}|+\cdots+|\mu_n^{(0)}|=1$.

Let $y_0=\mu_1^{(0)}x_1+\mu_2^{(0)}x_2+\cdots+\mu_n^{(0)}x_n$, then we obtain

$$||y_0-y^*_{k_{m_j}}||\leq|\mu_1^{(0)}-\mu_1^{(k_{m_j})}|^r\,||x_1||+\cdots$$
$$\cdots+|\mu_n^{(0)}-\mu_n^{(k_{m_j})}|^r\,||x_n||\to 0$$

as $m_j\to\infty$, that is, $y^*_{k_{m_j}}\to y_0$. Therefore, we have $y_0=0$, hence $\mu_i^{(0)}=0$ for any i.

This contradicts with the assumption.

Then, for all sums

$$|\lambda_1^{(k)}|^r+|\lambda_2^{(k)}|^r+\cdots+|\lambda_n^{(k)}|^r$$

and bounded sequence $\{\lambda_i^{(1)},\lambda_i^{(2)},\cdots\}$ there exists $\lambda_i^{(0)}$ such that $\lambda_i^{(k_j)}\to\lambda_i^{(0)}$ and

$$|\lambda_1^{(0)}|^r+|\lambda_2^{(0)}|^r+\cdots+|\lambda_n^{(0)}|^r$$

is bounded. Moreover

$$y_{k_j}\to y_0=\lambda_1^{(0)}x_1+\cdots+\lambda_n^{(0)}x_n$$

and y_0 is bounded.

Since $\{y_k\}$ is an arbitrary bounded sequence in L, any bounded set of a space with finite dimensional is compact.

Sufficient. Let L be a compact set. First, we select an element $x_1\in L$ such that $||x_1||=1$ and denote the linear space generated from x_1 by L_1. If $L=L_1$, then our theorem is proved. If $L\neq L_1$, then by the lemma we can select an element $x_2\in L$ such that $||x_2||=1$ and $||x_1-x_2||\geq\frac{1}{2}$. Let L_2 be the space generated from x_1 and x_2, then $L=L_2$ or $L\neq L_2$. If $L=L_2$, then it is obvious. If $L\neq L_2$, then we can inductively select an element $x_3\in L$ such that $||x_3||=1$, $||x_1-x_3||\geq\frac{1}{2}$ and $||x_2-x_3||\geq\frac{1}{2}$. If we continue this process, then we reduce two cases. For some n, we have $L=L_n$ and our theorem is proved. On the other hand, we have an infinite sequence $\{x_n\}$ such that $||x_n||=1$ and $||x_n-x_m||\geq\frac{1}{2}$ for $m\neq n$, then the set $\left\{x_n;\ ||x_n||=1 \text{ and } ||x_n-x_m||\geq\frac{1}{2}\right\}$ is not compact, and we have a contradiction. Thus L is a finite dimensional space.

Let E, F be two quasi-normed spaces with powers r, s and T a linear transformation which maps E into F.

Theorem II. *A linear transformation T is continuous if and* *only if there exists a positive number a for which the following inequality holds*:

$$\|T(x)\|_s \leq a\|x\|_r^{s/r}.$$

Proof. If T is continuous, then there exists a positive number B such that $\|T(x)-T(y)\|_s \leq 1$ whenever $\|x-y\|_r^{s/r} \leq B^{s/r}$. Thus $\|T(x)\|_s \leq 1$ whenever $\|x\|_r^{s/r} \leq B^{s/r}$, and for any $x \neq 0$,

$$\|T(x)\|_s = \left\|T\left(\frac{\|x\|_r^{1/r}}{B^{1/r}} \frac{B^{1/r}x}{\|x\|_r^{1/r}}\right)\right\|_s$$

$$= \left(\frac{\|x\|_r}{B}\right)^{s/r} \left\|T\left(\frac{B^{1/r}x}{\|x\|_r^{1/r}}\right)\right\|_s$$

$$\leq \left(\frac{\|x\|_r}{B}\right)^{s/r}.$$

If we denote $\left(\frac{1}{B}\right)^{s/r}$ by a, then a is a positive number which satisfies the inequality.

Conversely, we have

$$\|T(x)-T(y)\|_s = \|T(x-y)\|_s \leq a\|x-y\|_r^{s/r} < \varepsilon$$

whenever $\|x-y\|_r^{s/r} < \frac{\varepsilon}{a}$. Thus T is continuous.

Let $\mathcal{L}(E, F)$ be the set of continuous linear transformations which map E into F, these being two quasi-normed spaces with powers r, s. Let us observe that in $\mathcal{L}(E, F)$ the operations $T+S$, λT defined by $(T+S)(x) = T(x)+S(x)$, $(\lambda T)(x) = \lambda T(x)$ are well-defined, and $\mathcal{L}(E, F)$ is a linear space. We can define a quasi-norm of T in $\mathcal{L}(E, F)$ by taking the smallest number a such that $\|T(x)\|_s \leq a\|x\|_r^{s/r}$ for all $x\ (\neq 0)$. Hence we may denote

$$\|T\| = \sup_{\|x\|_r^{s/r} \leq 1} \|T(x)\|_s.$$

For, if $\|x\|_r^{s/r} \leq 1$, then we have

$$\|T(x)\|_s \leq \|T\|\ \|x\|_r^{s/r} \leq \|T\|,$$

and

$$\sup_{\|x\|_r^{1/r} \leq 1} \|T(x)\|_s \leq \|T\| \tag{1}$$

on the other hand, there exists an x' such that

$$\|T(x')\|_s > (\|T\|-\varepsilon)\|x'\|_r^{s/r}$$

for any ε. Let $x_1 = \frac{x'}{\|x'\|_r^{1/r}}$, then we have

$$\|T(x_1)\|_s = \frac{1}{\|x'\|_r^{s/r}} \|T(x')\|_s > \|T\|-\varepsilon.$$

Since $\|x_1\|^{s/r} = 1$,

$$\sup_{\|x\|_r^{s/r} \leq 1} \|T(x)\|_s \geq \|T(x_1)\|_s > \|T\|-\varepsilon. \tag{2}$$

By the arbitrariness of ε,

$$\sup_{\|x\|_r^{s/r} \leq 1} \|T(x)\|_s > \|T\|. \tag{3}$$

Hence we have

$$\sup_{\|x\|_r^{s/r}} \| T(x) \|_s = \| T \|.$$

This implies that $\mathcal{L}(E, F)$ is a quasi-normed space with the power s. s is the power of the range space F.

The space $\mathcal{L}(E, F)$ of all continuous linear transformations of a quasi-normed space E into a quasi-normed space F is a quasi-normed space. If F is a (QN)-space, then $\mathcal{L}(E, F)$ is a (QN)-space. If F is a Banach space, then $\mathcal{L}(E, F)$ is a Banach space.

References

[1] L. A. Ljusternik und W. Sobolew: Elemente der Funktionalanalysis Akademie-Verlag, Berlin.
[2] T. Konda: On quasi-normed space. I, Proc. Japan Acad., **35**, 340–342 (1959).
[3] M. Pavel: On quasi normed spaces, Bull. Acad. Sci., **5**, no. 5, 479–487 (1957).
[4] S. Rolewicz: On a certain class of linear metric spaces, Bull. Acad. Sci., **5**, no. 5, 471–473 (1957).