10. On a Problem of Royden on Quasiconformal Equivalence of Riemann Surfaces

By Mitsuru NAKAI
Mathematical Institute, Nagoya University

(Comm. by K. KUNUGI, M.J.A., Jan. 12, 1960)

1. **Definitions and problem.** We denote by $HBD(R)$ the totality of complex-valued bounded harmonic functions on a Riemann surface R with finite Dirichlet integrals. We use the following convention. If R is of null boundary, then the complex number field C is considered not to be contained in $HBD(R)$, that is, the constant function is not HBD-function and hence $HBD(R)$ is empty. On the other hand, if R is of positive boundary, then C is considered to be contained in $HBD(R)$.

Now consider the set $A(R)$ of all bounded and continuously differentiable functions on R with finite Dirichlet integrals. Then there exists a compact Hausdorff space $\tilde{R}$ containing R as its open dense subset and any function in $A(R)$ is continuously extended to $\tilde{R}$. Such a space $\tilde{R}$ is unique up to a homeomorphism fixing R. The set $\partial R=\tilde{R}-R$ is called the *ideal boundary* of R.

Let $\{R_n\}_{n=0}^{\infty}$ be an exhaustion of R with R_0=empty set. For each n, consider the family $\{F^{(n)}\}$ of closed subsets $F^{(n)}$ of $\tilde{R}-R_n$ such that any real-valued continuous function on $\tilde{R}-R_n$, which belongs to $HBD(R-\bar{R}_n)$, takes its maximum and minimum on $F^{(n)}$. The set

$$\bigcap_{n=0}^{\infty}\bigcap_{\{F^{(n)}\}} F^{(n)}$$

is empty or the compact subset of ∂R. We denote this set by $\partial_1 R$. Denote by $A_1(R)$ the totality of functions in $A(R)$ which vanish on $\partial_1 R$. Then any function f in $A(R)$ is decomposed into two parts u in $HBD(R)$ and $f-u$ in $A_1(R)$. This decomposition is unique and so we denote u by πf. Then it holds that

$$D[\pi f, f-\pi f]=\iint_R d(\pi f)\wedge {}^*\overline{d(f-\pi f)}=0.$$

Consider the following algebraic operations in $HBD(R)$: for arbitrary two functions u and v in $HBD(R)$ and for any complex number a, we define *addition*, *scalar multiplication* and *multiplication* by the following

$$(u+v)(p)=u(p)+v(p);$$
$$(au)(P)=a(u(p));$$
$$(u\times v)(p)=(\pi(uv))(p),$$

where p is an arbitrary point in R and $(uv)(p)=u(p)v(p)$.
With respect to these operations, $HBD(R)$ is an algebra over C. The algebra $HBD(R)$ can be *normed* by

$$||u||=||u||_\infty+\sqrt{D[u]},$$

where $||u||_\infty=\sup(|u(p)|; p\in R)$, and $HBD(R)$ becomes a *normed ring*.

We consider another topology, which will be called *BD-topology* defined by a convergence of sequences, where a sequence $\{f_n\}$ in $HBD(R)$ is said to converge to 0 if the sequence $\{||f_n||_\infty\}$ is bounded and $\{f_n\}$ converges to 0 uniformly on any compact subset of R and also $\{D[f_n]\}$ converges to 0.

H. L. Royden proved in his paper [2] the following interesting

Theorem. *If two Riemann surfaces R and R' are quasiconformally equivalent,*[1] *then $HBD(R)$ and $HBD(R')$ are algebraically isomorphic and this isomorphism is homeomorphic with respect to BD-topology.*[2]

And he raised a question whether the converse of the above holds or not. Precisely speaking, he raised the following question.

Problem. *If two topologically equivalent Riemann sufaces R and R' have BD-homeomorphic*[3] *and algebraically isomorphic algebras $HBD(R)$ and $HBD(R')$, then can we conclude that R and R' are quasiconformally equivalent?*

The aim of this note is to give *a negative answer* to this problem, that is, *there exist two topologically equivalent Riemann surfaces R and R' having BD-homeomorphic and algebraically isomorphic algebras $HBD(R)$ and $HBD(R')$ which are not quasiconformally equivalent.*

2. BD-topology in HBD(R). Fix a point p in R. We introduce *B_pD-topology* in $HBD(R)$. A sequence $\{u_n\}$ in $HBD(R)$ converges to 0 in B_pD-topology if the sequence $\{||u_n||_\infty\}$ is bounded and the sequences $\{u_n(p)\}$ and $\{D[u_n]\}$ converge to 0. First we state

Proposition 1. *Take an arbitrary point p in R. Then B_pD-topology coincides with BD-topology in $HBD(R)$.*

Proof. It is evident that the convergence in BD-topology implies the convergence in B_pD-topology. Thus we have only to prove that, if $\{u_n\}$ converges to 0 in B_pD-topology, then it converges to 0 in BD-topology.

Let r be a point in R. We associate a local parameter z with each r such that $z(r)=0$ and the parameter neighborhood of r is sent to $(|z|<2)$ by z. We denote by $\Delta(r)$ and $\Delta_1(r)$ the sets $(q; |z(q)|<1)$ and

1) This means that there exists a quasiconformal mapping of R onto R' in the sense of Pfluger-Ahlfors-Mori.

2) By the well-known Gelfand's theorem, this is also homeomorphic with respect to the norm $||f||$.

3) I.e. homeomorphic with respect to BD-topology.

$(q; |z(q)|<2)$ respectively. Then we get

$$|u_n(q)| \leqq |u_n(r)| + k\sqrt{D[u_n]}$$

for each q in $\Delta(r)$, where k is a universal constant independent of $\Delta(r)$.

Let R_0 be a set of points q in R such that $\{u_n\}$ converges to 0 uniformly on every $\Delta(q)$. R_0 is non-void since p belongs to R_0 by the the above inequality. By the same reason, R_0 is open and closed. Hence $R=R_0$, or $\{u_n\}$ converges to 0 in BD-topology.

3. Unicity of the multiplicative structure. We denote by $HB(R)$ the vector space of all complex-valued bounded harmonic functions on R. Here we consider that C is contained in $HB(R)$. If we use the term "*vector subspace*" X of $HB(R)$, we consider only the vector subspace X of $HB(R)$ such that X contains C and any u belongs to X if $\overline{u}$ is in X, where $\overline{u}(p)=\overline{u(p)}$ at every point p in R.

By a "*multiplicative structure*" in X, we mean that a commutative multiplication $u \odot v$, satisfying

$$||u \odot u||_\infty = ||u||_\infty^2,$$

is defined in X and X becomes a normed algebra. Now we can prove

Lemma. *There exists at most one multiplicative structure in any vector subspace X of $HB(R)$.*

Proof. Suppose that there exist two multiplicative structures which are defined by the multiplications $u \odot_1 v$ and $u \odot_2 v$, respectively. We denote these multiplicative structures by X_1 and X_2. Note that $X=X_1=X_2$ as the set. Without loss of generality, we may assume that X (and hence X_1 and X_2) is closed under the norm $||f||_\infty$. Hence we may consider X_i to be the totality $C(X_i^*)$ of complex-valued continuous functions on a compact Hausdorff space X_i^* by the Gelfand's representation theorem of a normed ring. We denote by u_i the element u belonging to X considered in X_i. Then $u_1 \to u_2$ gives an isometric linear isomorphism of X_1 onto X_2. Hence, by the theorem of Stone [3], $u_1 \to u_2$ is accomplished by an algebraic isomorphism of X_1 onto X_2 followed by a multiplication of a fixed function in X_2 with the modulus 1 at every point of X_2. But the constant function 1 in X_1 is sent to 1 in X_2, $u_1 \to u_2$ itself is an algebraic isomorphism. Thus $u \odot_1 v = u \odot_2 v$ for every pair of u and v in X. This completes the proof.

Now suppose that $R \notin O_G$. Then $HBD(R)$ is a vector subspace of $HB(R)$ in our sense. Hence we can conclude that

$u \times v$ is a unique multiplicative structure in $HBD(R)$.

From this, we get the following

Proposition 2. *If a linear isomorphism σ of $HBD(R)$ onto $HBD(R')$ possesses the property that $c^\sigma = c$ for all c in C and $||u^\sigma||_\infty = ||u||_\infty$, then σ is an algebraic isomorphism of $HBD(R)$ onto $HBD(R')$.*

4. Counter-example. Consider a Riemann surface R. If there exists a surface R' containing R as its subsurface and any HBD-function u defined on R can be extended to R' so as to belong to the class HBD no R'. Then we say that R' is an *HBD-continuation* of R. If there exists no HBD-continuation of R except itself, we say that R is *HBD-maximal.*

For an arbitrary surface R not belonging to the class O_G, there always exists by Zorn's lemma an HBD-continuation of R which is HBD-maximal. We denote it by R^*. Let $u^*\in HBD(R^*)$ be the continuation of $u\in HBD(R)$ to R^*. Then $i:u\to u^*$ gives a linear isomorphism of $HBD(R)$ onto $HBD(R^*)$ which preserves C. Moreover, it holds that $\|u\|_\infty=\|u^*\|_\infty$ and $D_R[u]=D_{R^*}[u^*]$. From these and from Propositions 1 and 2, we get

Proposition 3. *$HBD(R)$ and $HBD(R^*)$ are BD-homeomorphically and algebraically isomorphic.*

Example. Let R be $(|z|<1)$. We put $R_1=R-\{1/n\}_{n=2}^{\infty}\cup\{0\}$ and $R_2=R-\{1-1/n\}_{n=2}^{\infty}\cup\{0\}$ respectively. These surfaces R_1 and R_2 are topologically equivalent. Now we put $R_i^*=R$ $(i=1,2)$. Then the relation between R_i and R_i^* $(i=1,2)$ is the same as mentioned above. Hence we see that $HBD(R_1)$ and $HBD(R_2)$ are BD-homeomorphically and algebraically isomorphic. Assume that R_1 and R_2 are quasiconformally equivalent. Let t be a quasiconformal mapping of R_1 onto R_2. Since one point is deletable with respect to t, t is extended to a quasiconformal mapping of R onto itself such that $t(R-R_1)=R-R_2$. The set $R-R_1$ has a non-isolated point in R and $R-R_2$ has no non-isolated point in R. This is a contradiction. Thus this pair (R_1, R_2) of Riemann surfaces gives a negative solution to the problem of Royden stated in section 1.

Remarks. 1. In spite of the fact that the surfaces R_1 and R_2 mentioned above have BD-homeomorphic and algebraically isomorphic HBD-algebras, R_1 and R_2 are not quasiconformally equivalent. This comes from the fact that R_1 and R_2 are not HBD-maximal. Hence the problem of Royden must be confined with HBD-maximal surfaces.

2. Let R and R' be topologically equivalent. It is also an interesting question to settle whether $HBD(R)$ and $HBD(R')$ are isomorphic as algebra or not under the condition that $HBD(R)$ and $HBD(R')$ are isomorphic as vector space or furthermore homeomorphic with respect to the Dirichlet semi-norm (and so with respect to the HBD-norm).

3. It is known that *two Riemann surfaces R and R' are quasiconformally equivalent if and only if $A(R)$ and $A(R')$ are algebraically isomorphic* (cf. [1]). By the similar method as used in [1], we can prove that *R and R' are quasiconformally equivalent if and only if $A_1(R)$ and $A_1(R')$ are algebraically isomorphic.*

As the algebra $HBD(R)$ is the factor algebra $A(R)/A_1(R)$, the above example shows that $A(R)/A_1(R) \cong A(R')/A_1(R')$ (algebraically isomorphic) does not imply $A(R) \cong A(R')$ and $A_1(R) \cong A_1(R')$ even if R and R' are topologically equivalent.

References

[1] M. Nakai: Purely algebraic characterization of quasiconformality, Proc. Japan Acad., **35**, 440–443 (1959).
[2] H. L. Royden: A property of quasi-conformal mapping, Proc. Amer. Math. Soc., **5**, 266–269 (1954).
[3] M. H. Stone: Applications of the theory of Boolean rings to general topology, Trans. Amer. Math. Soc., **41**, 375–481 (1948).