2. On Ideals Defining Non-Singular Algebraic Varieties

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(Comm. by Z. SUETUNA, M.J.A., Jan. 12, 1960)

The purpose of this note is to prove the following

Theorem 1. *Let V be a non-singular irreducible algebraic variety of dimension d, defined over a field K in an affine space A^n. Then the ideal defining V over K is generated by at most $(n-d)(d+1)+1$ elements.*

To simplify our expression, we shall denote with $N_R(\mathfrak{a})$ the minimum number of elements generating the ideal $\mathfrak{a}$ in a Noetherian ring R. Our theorem means $N_R(\mathfrak{p})\leq(n-d)(d+1)+1$, when R is the polynomial ring of n variables $K[X_1, X_2,\cdots, X_n]$ over K and $\mathfrak{p}$ is the prime ideal defining V over K. Now R is a regular domain as defined e.g. in my former paper [2] and $R/\mathfrak{p}$ becomes also a regular domain as $\mathfrak{p}$ defines a non-singular variety. The rank of $\mathfrak{p}$ is $n-d$, as V is d dimensional (cf. [1]). So our Theorem 1 is contained in the following more general

Theorem 2. *Let R be a regular ring of dimension d and $\mathfrak{p}$ be a prime ideal of rank s in R such that $R/\mathfrak{p}$ is also a regular ring. Then* $N_R(\mathfrak{p})\leq s(d-s+1)+1$.

We shall begin with some lemmas.

Lemma 1. *Let R be a semi-local ring with maximal ideals $\mathfrak{m}_1, \mathfrak{m}_2,\cdots, \mathfrak{m}_s$ and $\mathfrak{a}=(a_1, a_2,\cdots, a_s)$ be any ideal generated by s elements in R. Then the simultaneous equations $x\equiv a_i$ (mod. $\mathfrak{a}\mathfrak{m}_i$) $1\leq i\leq s$, have a solution in R.*

Proof. Since R is semi-local, we have $R=(\bigcap_{j\neq i}\mathfrak{m}_j, \mathfrak{m}_i)$ for any i. So there exist elements e_i, d_i for any i such that $1=e_i+d_i$, where $e_i\notin\mathfrak{m}_i$, $\in\bigcap_{j\neq i}\mathfrak{m}_j$ and $d_i\in\mathfrak{m}_i$. Then, if we set $a=\sum_{i=1}^{s}e_ia_i$, this is a solution as is required.

Lemma 2. *Let R be a local ring with a maximal ideal $\mathfrak{m}$, $\mathfrak{a}$ be any ideal of R and $\mathfrak{b}$ be an ideal of R contained in $\mathfrak{a}$. Then, if $\mathfrak{a}=\mathfrak{b}+\mathfrak{a}\mathfrak{m}$, we have $\mathfrak{a}=\mathfrak{b}$.*

Proof. Set $\bar{R}=R/\mathfrak{b}$, $\bar{\mathfrak{a}}=\mathfrak{a}/\mathfrak{b}$ and $\bar{\mathfrak{m}}=\mathfrak{m}/\mathfrak{b}$. Then, by our assumption, we have $\bar{\mathfrak{m}}\bar{\mathfrak{a}}=\bar{\mathfrak{a}}$. So, $\bar{\mathfrak{a}}\subset\bigcap_{k=1}^{\infty}\bar{\mathfrak{m}}^k$. Since $\bar{R}$ is a local ring, we have, by Krull's theorem, $\bigcap_{k=1}^{\infty}\bar{\mathfrak{m}}^k=(0)$. That is, $\bar{\mathfrak{a}}=(0)$. This shows $\mathfrak{a}=\mathfrak{b}$.

Lemma 3. *Let R be a Noetherian ring and $\mathfrak{a}, \mathfrak{b}$ be two ideals of R such that $\mathfrak{a}\supset\mathfrak{b}$. If $\mathfrak{a}R_{\mathfrak{m}}=\mathfrak{b}R_{\mathfrak{m}}$ for any maximal ideal $\mathfrak{m}$ of R, we have $\mathfrak{a}=\mathfrak{b}$.*

Proof. Let a be any element of $\mathfrak{a}$ and put $\mathfrak{b}=(b_1, b_2, \cdots, b_t)$. By our assumption we have $\bar{a}=\alpha_1\bar{b}_1+\alpha_2\bar{b}_2+\cdots+\alpha_t\bar{b}_t$ in any R_m, where α_i is in R_m for any i and $\bar{a}, \bar{b}$ are the residues of a, b in R, respectively. Therefore we obtain $sa=r_1b_1+r_2b_2+\cdots+r_tb_t$, $r_i, s\in R$, $s\notin \mathfrak{m}$. Accordingly, if we set $\mathfrak{c}=\{c; ca\in\mathfrak{b}, c\in R\}$, then an ideal $\mathfrak{c}$ must coincide with R itself. This shows $\mathfrak{a}=\mathfrak{b}$.

From these lemmas we obtain the following

Proposition 1. *Let R be a semi-local ring with maximal ideals $\mathfrak{m}_1, \mathfrak{m}_2, \cdots, \mathfrak{m}_s$ and $\mathfrak{a}$ be any ideal of R. Then we have*

$$\mathrm{N}_R(\mathfrak{a})=\max_{1\le i\le s} \mathrm{N}_{R_{\mathfrak{m}_i}}(\mathfrak{a}R_{\mathfrak{m}_i}).$$

Proof. For any i we choose n_i elements $a_{i,1}, \cdots, a_{i,n_i}$ from $\mathfrak{a}$ such that $\mathfrak{a}R_{\mathfrak{m}_i}=(a_{i,1}, \cdots, a_{i,n_i})R_{\mathfrak{m}_i}$, where $n_i=\mathrm{N}_{R_{\mathfrak{m}_i}}(\mathfrak{a}R_{\mathfrak{m}_i})$. Set $n=\max_{1\le i\le s} n_i =\max_{1\le i\le s} \mathrm{N}_{R_{\mathfrak{m}_i}}(\mathfrak{a}R_{\mathfrak{m}_i})$. If $n_i<n$, we put $0=a_{i,n_i+1}=\cdots=a_{i,n}$. Consider a set of elements $(a_{1,j}, a_{2,j}, \cdots, a_{s,j})$ for any $1\le j\le n$ and apply Lemma 1 to this. Then, for any j, the simultaneous equations $x\equiv a_{i,j}$ (mod. $\mathfrak{a}\mathfrak{m}_i$) have a solution a_j in $\mathfrak{a}$. So we have

$$\mathfrak{a}R_{\mathfrak{m}_i}=(\bar{a}_{i,1}, \cdots, \bar{a}_{i,n})+\mathfrak{a}\mathfrak{m}_iR_{\mathfrak{m}_i}=(\bar{a}_1, \cdots, \bar{a}_n)+\mathfrak{a}\mathfrak{m}_iR_{\mathfrak{m}_i}.$$

Now if we put $\mathfrak{b}=(a_1, a_2, \cdots, a_n)$, we obtain $\mathfrak{a}R_{\mathfrak{m}_i}=\mathfrak{b}R_{\mathfrak{m}_i}$, by Lemma 2, for any i. Hence, according to Lemma 3, we have $\mathfrak{a}=\mathfrak{b}$. This shows $\mathrm{N}_R(\mathfrak{a})\le\max_{1\le i\le s} \mathrm{N}_{R_{\mathfrak{m}_i}}(\mathfrak{a}R_{\mathfrak{m}_i})$. The inverse inequality is obvious. So our assertion is proved.

By applying this proposition to a general Noetherian ring, we obtain

Proposition 2. *Let R be a d dimensional Noetherian ring and $\mathfrak{p}$ be a prime ideal of rank s in R. Set $k=\sup_{\mathfrak{m}} \mathrm{N}_{R_\mathfrak{m}}(\mathfrak{p}R_\mathfrak{m})$ where $\mathfrak{m}$ runs over all maximal ideals of R. Then we have* $\mathrm{N}_R(\mathfrak{p})\le k(d-s+1)+1$.

Remark. This proposition holds also if we replace the prime ideal $\mathfrak{p}$ by any ideal $\mathfrak{a}$. The proof runs similarly when $\mathfrak{a}$ is primary. For general $\mathfrak{a}$ the proof can be easily given in using the decomposition of $\mathfrak{a}$ in primary ideals.

Proof. By our assumption, if we choose suitably k elements $p_1^{(1)}, p_2^{(1)}, \cdots, p_k^{(1)}$ from $\mathfrak{p}$, we have $(p_1^{(1)}, p_2^{(1)}, \cdots, p_k^{(1)})=\mathfrak{p}\cap\mathfrak{a}_0$, where $\mathfrak{a}_0$ is an ideal of rank s or more. If rank $\mathfrak{a}_0>s$ we set $\mathfrak{a}_1=\mathfrak{a}_0$ and $\mathfrak{p}^{(0)}=0$. On the other hand, if rank $\mathfrak{a}_0=s$, we select an element $p^{(0)}$ from $\mathfrak{p}$ which is not contained in any prime ideal of rank s belonging to $\mathfrak{a}_0$. Then, in both cases, we have

$$(p^{(0)}, p_1^{(1)}, \cdots, p_k^{(1)})=\mathfrak{p}\cap\mathfrak{a}_1,$$

where rank $\mathfrak{a}_1\ge s+1$. If rank $\mathfrak{a}_1>s+1$, we set $\mathfrak{a}_2=\mathfrak{a}_1$ and $p_i^{(2)}=0$ for $1\le i\le k$. In case rank $\mathfrak{a}_1=s+1$, we denote by $\mathfrak{p}_1, \mathfrak{p}_2, \cdots, \mathfrak{p}_t$ all prime ideals of rank $s+1$ belonging to $\mathfrak{a}_1$. Set $R'=R_S$, where $S=\left\{s; s\in\bigcap_{i=1}^{t}(R-\mathfrak{p}_i)\right\}$. Then R' is a semi-local ring, hence, by applying Proposition 1 to R', $\mathfrak{p}$, we can choose k elements of $\mathfrak{p}$ such that

$$(p_1^{(2)}, p_2^{(2)}, \cdots, p_k^{(2)})R'=\mathfrak{p}R'.$$

So we have $$(p^{(0)}, p_1^{(1)}, \cdots, p_k^{(1)}, p_1^{(2)}, \cdots, p_k^{(2)})=\mathfrak{p}\cap\mathfrak{a}_2$$
in both cases, where rank $\mathfrak{a}_2\geq s+2$. By repeating this procedure we obtain similarly for any integer t

$$(p^{(0)}, p_1^{(1)}, \cdots, p_k^{(1)}, p_1^{(2)}, \cdots, p_1^{(t)}, \cdots, p_k^{(t)})=\mathfrak{p}\cap\mathfrak{a}_t$$

where rank $\mathfrak{a}_t\geq s+t$. However, since R is d dimensional, we have $\mathfrak{a}_t=R$ for any t such that $t\geq d-s+1$. Therefore this procedure can not be repeated more than $d-s+1$ times. Thus we obtain

$$\mathfrak{p}=(p^{(0)}, p_1^{(1)}, \cdots, p_k^{(1)}, \cdots, p_1^{(d-s+1)}, \cdots, p_k^{(d-s+1)}).$$

This shows that $\mathrm{N}_R(\mathfrak{p})\leq k(d-s+1)+1$.

The proof of Theorem 2 can be now carried out very easily. Let $\mathfrak{m}$ be any maximal ideal of R. Since $R_{\mathfrak{m}}$ and $(R/\mathfrak{p})_{\mathfrak{m}/\mathfrak{p}}$ are then regular local rings, $\mathfrak{p}R_{\mathfrak{m}}$ is generated by s elements in R, by a well-known result in the theory of local rings (cf. [1]). From Proposition 2 follows therefore $\mathrm{N}_R(\mathfrak{p})\leq s(d-s+1)+1$.

References

[1] Y. Akizuki: Theory of local rings, Lecture notes in Univ. Chicago (1958).
[2] S. Endo: On regular rings, Jour. Math. Soc. Japan, **11**, 159–170 (1959).