# 35. An Application of a Compact Normal Operator in Hilbert Spaces to the Theory of Functions

By Sakuji INOUE
Faculty of Education, Kumamoto University
(Comm. by K. KUNUGI, M.J.A., March 12, 1960)

In this paper we shall discuss the integration of a given function of a complex variable along a closed Jordan curve which encloses its denumerably infinite set of poles and its essential singularities, by making use of the properties of a compact normal operator in an abstract Hilbert space $\mathfrak{H}$ and of linear functionals with domain $\mathfrak{H}$.

Theorem 1. Let $f(\lambda)$ be holomorphic at all points of the closure $\overline{D}$ of a simply connected domain $D$ in the complex $\lambda$-plane, except at its poles $\{\lambda_n\}\in D$ tending to the point $\lambda=0$ interior to $D$ and at its non-isolated essential singularity $\lambda=0$.

If the principal part of the expansion of $f(\lambda)$ at any pole $\lambda_n$ is given by $\dfrac{\alpha_n}{\lambda-\lambda_n}$ and if $\sum_{n=1}^{\infty}|\alpha_n|<\infty$, then

$$\frac{1}{2\pi i}\int_{\partial D} f(\lambda)d\lambda=\sum_{n=1}^{\infty}\alpha_n,$$

where the complex curvilinear integration along the boundary $\partial D$ of $D$ is taken in the positive (anti-clockwise) direction.

Proof. Let $\{\varphi_n\}$ be an arbitrary complete orthonormal system in the abstract complex Hilbert space $\mathfrak{H}$ which is complete, separable and infinite dimensional, and let $E_n$ be the orthogonal projection of $\mathfrak{H}$ onto the subspace determined by $\varphi_n$.

If we now define $N$ by $N=\sum_{n=1}^{\infty}\lambda_n E_n$, it is easily verified that $N$ has the following properties:

1° the convergence of $\sum_{n=1}^{\infty}\lambda_n E_n$ is uniform, that is, $\left\|N-\sum_{n=1}^{p}\lambda_n E_n\right\|\to 0,\ (p\to\infty)$;

2° $\{\lambda_n\}$ is the point spectrum of $N$, and $E_n$ is the characteristic projection of $N$ corresponding to $\lambda_n$, $n=1,2,3,\cdots$;

3° $N$ is a compact normal operator in $\mathfrak{H}$ [1].

Since every linear continuous functional $L(y)$ on $\mathfrak{H}$ can be put in the form $L(y)=(y,x)$ where the generating element $x\in\mathfrak{H}$ is uniquely determined by the functional $L$ [4], from now on we shall denote by $L_x$ the functional $L$ associated with $x$.

Next we put

$$x=\sum_{n=1}^{\infty}\sqrt{\alpha_n}\,\varphi_n,\qquad \tilde{x}=\sum_{n=1}^{\infty}\sqrt{\bar{\alpha}_n}\,\varphi_n\qquad ((\sqrt{\alpha_n}\,\varphi_n,\ \sqrt{\bar{\alpha}_n}\,\varphi_n)=\alpha_n)$$

and consider the function $H(\lambda)$ defined by

$$H(\lambda)=f(\lambda)-L_{\tilde{x}}[(\lambda I-N)^{-1}x].$$

Then, it is first clear that $x$ and $\tilde{x}$ both belong to $\mathfrak{H}$ by virtue of the assumption on $|\alpha_n|$ for $n=1,2,3,\cdots$, and that $H(\lambda)$ can be holomorphic on $\bar{D}$ on account of the facts that all points in the complex plane, except $0, \lambda_1, \lambda_2, \cdots$, belong to the resolvent set $\rho(N)$ of $N$ and hence

$$(\lambda I-N)^{-1}=\int_G \frac{1}{\lambda-z}\,dE(z), \qquad \lambda\in\rho(N)$$
$$=\sum_{n=1}^{\infty}\frac{E_n}{\lambda-\lambda_n} \qquad [2]$$

where $G$ and $\{E(z)\}$ denote the complex plane and the spectral family associated with $N$ respectively. According to the Cauchy theorem on curvilinear integral, we have therefore

$$\int_{\partial D} H(\lambda)d\lambda=0;$$

and this result permits us to conclude that

$$\frac{1}{2\pi i}\int_{\partial D} f(\lambda)d\lambda=\frac{1}{2\pi i}\int_{\partial D} L_{\tilde{x}}[(\lambda I-N)^{-1}x]d\lambda$$
$$=L_{\tilde{x}}\left[\frac{1}{2\pi i}\int_{\partial D}(\lambda I-N)^{-1}d\lambda\cdot x\right]$$
$$=L_{\tilde{x}}(Ix) \qquad [3]$$
$$=\sum_{n=1}^{\infty}\alpha_n.$$

The theorem has thus been proved.

Corollary 1. In the case where the sequence $\{\lambda_n\}$ converges to a non-zero complex number, the same assertion as that stated in Theorem 1 is also valid.

Proof. Let $\lambda_0$ be the limiting point of $\{\lambda_n\}$, and let $\{\bar{E}(z)\}$ denote the spectral family of the compact normal operator $\bar{N}$ defined by $\bar{N}=\sum_{n=1}^{\infty}(\lambda_n-\lambda_0)E_n$ where $E_n$ has the same meaning as before. Then, since the characteristic projection of $\bar{N}$ corresponding to the characteristic value $\lambda_n-\lambda_0$ is identical with $E_n$ for any positive integer $n$, we have, for every $\lambda$ different from $\lambda_n$, $n=0,1,2,\cdots$,

$$[(\lambda-\lambda_0)I-\bar{N})]^{-1}=\int_G \frac{1}{\lambda-\lambda_0-z}\,d\bar{E}(z)$$
$$=\sum_{n=1}^{\infty}\frac{E_n}{\lambda-\lambda_0-(\lambda_n-\lambda_0)}$$
$$=\sum_{n=1}^{\infty}\frac{E_n}{\lambda-\lambda_n},$$

which implies that $f(\lambda)-L_{\tilde{x}}[((\lambda-\lambda_0)I-\bar{N})^{-1}x]$ is holomorphic at every point $\lambda$ on $\bar{D}$. In consequence, by the same reasoning as that used in the proof of the preceding theorem, it is easily verified that the present corollary holds.

Theorem 2. Let $f(\lambda)$ be subject to the hypotheses of Theorem 1. If the principal part of the expansion of $f(\lambda)$ at any pole $\lambda_n$ is given by

$$\sum_{\nu=1}^{p(n)} \frac{\alpha_\nu^{(n)}}{(\lambda-\lambda_n)^\nu},$$

where $p(n)$ denotes the order of the pole $\lambda_n$, and if $\sum_{n=1}^{\infty} |\alpha_\nu^{(n)}| < \infty$ for every admissible value of $\nu$ under the condition that $\alpha_\nu^{(n)}=0$ for $\nu > p(n)$, then

$$\frac{1}{2\pi i}\int_{\partial D} f(\lambda)d\lambda = \sum_{n=1}^{\infty} \alpha_1^{(n)},$$

where $\partial D$ is the boundary of $D$, positively oriented.

Proof. If we put

$$x_\nu = \sum_{n=1}^{\infty} \sqrt{\alpha_\nu^{(n)}}\,\varphi_n, \quad \tilde{x}_\nu = \sum_{n=1}^{\infty} \sqrt{\bar{\alpha}_\nu^{(n)}}\,\varphi_n \quad ((\sqrt{\alpha_\nu^{(n)}}\,\varphi_n, \sqrt{\bar{\alpha}_\nu^{(n)}}\,\varphi_n) = \alpha_\nu^{(n)})$$

where $\{\varphi_n\}$ is the same symbol as that used before, then, by the assumption concerning $|\alpha_\nu^{(n)}|$, $x_\nu$ and $\tilde{x}_\nu$ belong to $\mathfrak{H}$ for every admissible value of $\nu$. Since, for any point $\lambda$ in the resolvent set $\rho(N)$ of the compact normal operator $N$ defined at the beginning of the proof of Theorem 1,

$$(\lambda I-N)^{-\nu} = \int_G \frac{1}{(\lambda-z)^\nu}\,dE(z) = \sum_{n=1}^{\infty} \frac{E_n}{(\lambda-\lambda_n)^\nu},$$

we see that $f(\lambda)-\sum_{\nu\leqq M} L_{\tilde{x}_\nu}[(\lambda I-N)^{-\nu}x_\nu]$, where $M=\max\{p(1), p(2), \cdots\}$, is holomorphic on $\bar{D}$, and hence that

$$(1) \qquad \frac{1}{2\pi i}\int_{\partial D} f(\lambda)d\lambda = \sum_{\nu\leqq M} \frac{1}{2\pi i}\int_{\partial D} L_{\tilde{x}_\nu}[(\lambda I-N)^{-\nu}x_\nu]d\lambda.$$

In addition, applying the fact that $(\lambda I-N)^{-1}$ has derivatives of all orders, with

$$\frac{d^n}{d\lambda^n}(\lambda I-N)^{-1} = (-1)^n n!\,(\lambda I-N)^{-(n+1)}, \quad (\lambda\in\rho(N);\ n=1, 2, \cdots)\ [5],$$

there is no difficulty in showing that

$$\frac{1}{2\pi i}\int_{\partial D}(\lambda I-N)^{-\nu}d\lambda = -\frac{1}{2(\nu-1)\pi i}\int_{\partial D} d(\lambda I-N)^{-\nu+1}, \quad \nu=2, 3, \cdots, M,$$

$$(2) \qquad = \mathbf{0},$$

where $\mathbf{0}$ denotes the null operator.

By applying (2) to (1) we find that

$$\frac{1}{2\pi i}\int_{\partial D} f(\lambda)d\lambda = L_{\tilde{x}_1}\Big[\frac{1}{2\pi i}\int_{\partial D}(\lambda I-N)^{-1}d\lambda\cdot x_1\Big]$$
$$=L_{\tilde{x}_1}(Ix_1)$$
$$=\sum_{n=1}^{\infty}\alpha_1^{(n)},$$

as we wished to prove.

Corollary 2. In the case where $\lambda_n$ converges to a non-zero complex number $\lambda_0$ when $n$ becomes infinite, the result stated in Theorem 2 is also valid.

Proof. Let $\bar{N}$ denote the compact normal operator $\sum_{n=1}^{\infty}(\lambda_n-\lambda_0)E_n$ as before. Then, by applying $[(\lambda-\lambda_0)I-\bar{N}]^{-\nu}$ in place of $(\lambda I-N)^{-\nu}$ employed in the proof of Theorem 2, the present corollary can be established in the same way as Theorem 2 was proved.

Theorem 3. Let $\{\lambda_n\}_{n=1,2,3,\cdots}$ be all poles of a function $f(\lambda)$ defined on the closure $\bar{D}$ of a simply connected domain $D$ in the complex $\lambda$-plane; let $c_1, c_2,\cdots, c_M$ be all accumulation points of $\{\lambda_n\}$; let $\{\lambda_{(\kappa,n)}\}_{n=1,2,3,\cdots}$ be a subsequence of $\{\lambda_n\}$, which converges to $c_\kappa$; let $\{\lambda_{(\kappa,n)}\}_{\substack{\kappa=1,2,\cdots,M\\ n=1,2,3\cdots}}$ contain all elements of $\{\lambda_n\}$; let $\{\lambda_n\}$ and $c_1, c_2,\cdots, c_M$ be in the interior of $D$; and let $f(\lambda)$ be holomorphic at all points of $\bar{D}$, except at those poles and non-isolated essential singularities.

If the principal part of the expansion of $f(\lambda)$ at any pole $\lambda_{(\kappa,n)}$ is given by

$$\sum_{\nu=1}^{p(\kappa,n)}\frac{\alpha_\nu^{(\kappa,n)}}{(\lambda-\lambda_{(\kappa,n)})^\nu},$$

where $p(\kappa, n)$ is the order of the pole $\lambda_{(\kappa,n)}$, and if $\sum_{n=1}^{\infty}|\alpha_\nu^{(\kappa,n)}|$ converges for all admissible values of $\nu$ under the condition that $\alpha_\nu^{(\kappa,n)}=0$ for $\nu>p(\kappa, n)$, then

$$\frac{1}{2\pi i}\int_{\partial D} f(\lambda)d\lambda=\sum_{\kappa=1}^{M}\sum_{n=1}^{\infty}\alpha_1^{(\kappa,n)},$$

where $\partial D$ is the boundary of $D$, positively oriented.

Proof. In $D$ we can first construct disjoint, simply connected domains $D_1, D_2,\cdots, D_M$ such that the poles $\{\lambda_{(\kappa,n)}\}_{n=1,2,3,\ldots}$, together with the corresponding essential singularity $c_\kappa$, are in the interior of $D_\kappa$ for $\kappa=1, 2,\cdots, M$. Then, by virtue of the application of Corollary 2 to $f(\lambda)$ restricted on the closure $\bar{D}_\kappa$ of $D_\kappa$, we obtain

$$\frac{1}{2\pi i}\int_{\partial D_\kappa} f(\lambda)d\lambda=\sum_{n=1}^{\infty}\alpha_1^{(\kappa,n)},$$

where $\partial D_\kappa$ denotes the boundary of $D_\kappa$, positively oriented.

On the other hand, since $f(\lambda)$ is holomorphic on the closed domain $R$ bounded by the boundaries of $D$ and $D_\kappa$, $\kappa=1, 2,\cdots, M$, the integral of $f(\lambda)$ along the boundary of $R$ is equal to zero according to Cauchy's curvilinear integral theorem.

In consequence, we have

$$\frac{1}{2\pi i}\int_{\partial D} f(\lambda)d\lambda=\sum_{\kappa=1}^{M}\frac{1}{2\pi i}\int_{\partial D_\kappa} f(\lambda)d\lambda$$
$$=\sum_{\kappa=1}^{M}\sum_{n=1}^{\infty}\alpha_1^{(\kappa,n)},$$

as we were to prove.

Remark. From the extension of Mittag-Leffler's theorem on the decomposition of meromorphic functions into simple fractions, it is clear that there exist such functions $f(\lambda)$ as we have treated above.

## References

[1] S. Inoue: Normal operators in Hilbert spaces and their applications, Mem. Fac. Educ., Kumamoto Univ., **7**, Supp. no. 2, 2 (1959).
[2] ——: Ibid., 22-24.
[3] ——: Ibid., 21.
[4] Angus E. Taylor: Introduction to Functional Analysis, New York, 245 (1958).
[5] ——: Ibid., 257-258.